19 Jan 2009 **Algebraic Lie Theory at the Newton Institute**

D. Vogan **Representations of real reductive Lie groups I**

Representations = unitary, irreducible

$G(\mathbb{R})$ ______

$\updownarrow$

geometry on flag varieties ______

$\updownarrow$

Hecke algebras, ______
algebra, KL-polynomials etc.

$\mathbb{R}$ - not algebraically closed!
G - topological group.

unitary representation: $\pi : G \to$ unitary linear operators on a Hilbert space V

So that $\pi : G \times V \longrightarrow V$
$(g, v) \longmapsto \pi(g)v$ is continuous

Hilbert space: V endowed w/ positive Hermitian form $\langle , \rangle$
unitary operators $\pi(g)$ preserve the form.

Formally: Hermitian form $\longleftrightarrow$ map: $V \longrightarrow$ some kind of dual space

~~doesn't~~
<u>Flag varieties</u>

(New) V (basic example of a reductive group)

finite dim vector space ~~over k~~ of dim n over k.

<u>complete flags in V</u> = $\mathcal{F}(V) = \{$chains of subspaces: $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, $\dim V_i = i\}$

$\mathcal{F}(V)$ = projective algebraic variety defined over k. $\dim \mathcal{F}(V) = \dfrac{n(n-1)}{2}$

$GL(V)$ acts transitively on $\mathcal{F}(V)$

geometry underlying the representation theory of $GL(V)$.

<u>version</u> for any <u>reductive</u> G:

say k - algebraically closed

$\mathcal{B}$ = variety of all Borel subgroups of G, projective algebraic variety. G acts transitively on $\mathcal{B}$.

want to understand this geometry.

$GL(V)$ case: have partial flag varieties
$\longleftrightarrow$ any subset A of $\{1,\dots,n-1\}$

$\mathcal{F}_A(V)$ = chains of subspaces
$$0 = V_0 \subset \cdots \subset V_n = V$$
w/ V_i ~~omi~~ omitted if $i \in A$.

$\underbrace{\mathcal{F}(V) \longrightarrow \mathcal{F}_A(V)}$ forget V_i, $i \in A$

projective G-equivariant map.
In fact a fibration w/ fiber over a partial flag = product of complete flag varieties in subquotient vector spaces.

eg $n = 4$ $A = \{1,3\}$

$$\mathcal{F}_A(V) = \{ V_0 \subset V_2 \subset V_4 = V \mid \dim V_i = i \}$$

= Grassmann variety of 2-planes in 4-dimensional V.

fiber over flag: (lines $V_1 \subset V_2$) $\times$ (3-dim V_3, $V_2 \subset V_3 \subset V_4$)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$
line V_3/V_2 in 2-dim V_4/V_2

$$\simeq \mathbb{P}^1 \times \mathbb{P}^1$$

$\mathcal{F}(V)$ fibers over $\mathcal{F}_A(V)$, w/ fiber $\mathbb{P} \times \mathbb{P}^1$

S = G-conjugacy classes of subgroups minimally properly containing Borel subgroups.

"simple roots" in G.

~~Thm~~ Thm [conjugacy classes of subgroups containing Borel subgroups]
$\longleftrightarrow$ [subset $A \subset S$]

P_A = variety of subgroups of type A: projective algebraic G-space

($s \in S$) $B \to \cancel{P} P_s$ is a $\mathbb{P}^1$-bundle.

Idea Hecke algebras $\longleftrightarrow$ geometry of various $\mathbb{P}^1$-fibrations $B \to P_s$ ($s \in S$).

Goal
How do we use these fibrations to control the representation theory of G.

eg $GL(n, \mathbb{R}) \rightsquigarrow$ n-dimensional V over $\mathbb{C}$

choice of orthogonal form on V (over $\mathbb{C}$).
study (complex) orthogonal $O(V)$ acting on
$\mathcal{F}$ = complete flags in V.

Flag: $0 \subset V_1 \subset \cdots \subset V_n = V$, fixed orthogonal group

orthogonal flag: $0 \subset V_1^{\perp} \subset \cdots \subset V_n^{\perp} = V$

$\perp$ - orthogonal w/ respect to $\cancel{\langle,\rangle}$ orthogonal form

Two flags in n-dim space (up to $GL(V)$ action) $\longleftrightarrow$ permutation of $\{1, \dots, n\}$

$\underbrace{\text{"relative position"}}_{\text{involution in } S_n}$

our case: how singular is the form restricted to V_i.

As i increases $V_i/\text{radical of } \langle \cdot,\cdot \rangle|_{V_i}$

has to grow in dimension.

__Prop__ orbits of $O(V)$ on $F(V) \leftrightarrow$ involutions in S_n.

$\underbrace{\text{Maximally isotropic flags}}_{V_1, \dots, V_{\lfloor n/2 \rfloor} \text{ isotropic, remaining are orthogonal}} \longleftrightarrow$ identity

__Beilinson - Bernstein__ (~ 1980)

characters of irred. reps of $GL(n, \mathbb{R})$ (∞-dim) $\longleftrightarrow$ cohomology of $O(V)$ equivariant perverse sheaves on $F(V)$. $\dim V = n$.

(cf. P. Achar's lectures): parametrize irred. perverse sheaves by

- orbit of $O(V)$ on $F(V)$.
- local systems on orbits (will explain later)

$\rightsquigarrow$ generalized KL-algorithm to compute cohomology

How do we add "unitary" into this mix?

Where does $O(V)$ come from?

k - non algebraically closed

$\bar{k}$ - algebraic closure of k.

Assume $\bar{k}/k$ is a Galois extension, group $\Gamma = \mathrm{Gal}(\bar{k}/k)$

Given: $X/\bar{k}$; defining X over $k \leftrightarrow$ making Γ act on $X(\bar{k})$ w/ Γ-action on $\bar{k}$ twisting everything (not algebraic!).

eg $k = \mathbb{R}$, need action on $X(\mathbb{C})$ acts as local coord. algebras w/ $z \to \bar{z}$ twist.

<u>NOT</u> amenable to alg geom/alg closed field

<u>BIG EXCEPTION</u>: $k = \mathbb{F}_q$; Galois action $\leftrightarrow z \to z^q$ twist

can study by algebraic geometry.

Reductive alg. groups over $\mathbb{R}$: E. Cartan "similar" machinery to make everything algebraic...

$\hookrightarrow$ G-reductive alg/$\mathbb{C}$; try to define it over $\mathbb{R}$. Look for $\sigma: G(\mathbb{C}) \to G(\mathbb{C})$, antiholomorphic, order 2, group hom, NOT in $\mathrm{Aut}_{\text{Alg. grp}}(G(\mathbb{C}))$.

Cartan: There's distinguished class of such automorphism $\{\sigma_0\}$.
$G(\mathbb{C})^{\sigma_0} \underset{def}{=} G(\mathbb{R}, \sigma_0)$ is compact.

<u>Thm</u> (Cartan) given real form σ, there is a distinguished real form $\{\sigma_0\}$ s.t. ~~σ_0 compact~~ it is compact and $\sigma\sigma_0 = \sigma_0\sigma$ (as antiholomorphic auts. of $G(\mathbb{C})$)

Hence $\theta \underset{def}{=} \sigma \cdot \sigma_0$ is an algebraic aut. of order 2 of $G(\mathbb{C})$.

Makes bijection {real forms of $G(\mathbb{C})$} upto $G(\mathbb{C})$ conjugation $\longleftrightarrow$ elements of order 2 in $\mathrm{Aut}_{alg.\,grp}(G(\mathbb{C})) / G(\mathbb{C})$-conjugation

Real forms of $G(\mathbb{C}) / \sim$ $\longleftrightarrow$ $\{$elements of order 2 in Aut $G(\mathbb{C})\} / \sim$

$GL(n, \mathbb{C})$
$\cup$
$GL(n, \mathbb{R})$ $\longleftrightarrow$ inverse transposition automorphism of $GL(n, \mathbb{C})$