~~W~~ 20 Jan 2009 — Algebraic Lie Theory at the Newton Institute

D. Vogan — Representations of Real Reductive Lie groups II

1) Describe beginning of path:
(reps. of Lie group over $\mathbb{R}$) $\rightsquigarrow$ algebra

2) Say more about Cartan bijection

$\left(\begin{array}{l}\text{real forms of}\\ \text{complex red. grp}\end{array}\right) \longleftrightarrow \left(\begin{array}{l}\text{algebraic automorphis-}\\ \text{-ms of order 2 of complex}\\ \text{reductive group}\end{array}\right)$

G = real Lie group

unitary representation: $\pi : G \longrightarrow$ unitary operato-
-rs on a Hilbert space.

continuity: $G \times V \longrightarrow V$ continuous.

V ↑ Hilbert space: metric space from norm

π is called irreducible if has exactly two ~~invariant~~ closed invariant subspaces: $0, V$.

eg. $G = SL(2, \mathbb{R})$ = 2×2 real matrices w/ det 1.
$X = \mathbb{R}\mathbb{P}^1$ = lines in $\mathbb{R}^2$ through origin
$\simeq$ unit circle in $\mathbb{R}^2/\pm 1$

Hilbert space: V = square integrable half-
-densities on X
$\sim$ functions $f(\theta)$ on circle in

plane s.t. $f(\theta) = f(-\theta)$ and

$$\int_0^{2\pi} |f(\theta)|^2 d\theta < \infty$$

This is in fact an irreducible unitary rep. of G.

V has lots of invariant subspaces.
- bounded functions f
- smooth functions f
- real analytic f

} all dense in V

Harish-Chandra: how to algebraize V:

Take $SO(2) \subset SL(2, \mathbb{R})$

$\uparrow$ rotations of the circle

$V \supset V^K =$ "K-finite" L^2 half densities on circle

= half densities whose rotational translates span a finite dimensional space

$$= \left\{ \sum_{m=-N}^{N} a_m e^{im\theta} |d\theta|^{1/2} \;\middle|\; \begin{array}{l} a_m \in \mathbb{C} \\ a_m = 0 \text{ if } m\text{-odd} \end{array} \right\}$$

$$= \operatorname{span} \{ e^{im\theta} |d\theta|^{1/2} \mid m\text{-even} \}$$

~~Harish-C~~

Def (back to general unitary π of Lie group G)

A vector $v \in V$ is called smooth if the map $G \to V$ $g \mapsto \pi(g)v$ is C^∞.

$V^\infty =_{\text{def}}$ space of smooth vectors in V

(Gårding) V^∞ is dense in V; V^∞ carries a natural complex representation of $\mathfrak{g} = \mathrm{Lie}(G)$

(Hilbert space is a complex vector space)

<u>Thm</u> (first for SL(2) example) $V^K \subset V^\infty$ is invariant subspace for Lie algebra representation; get representations of
- Lie algebra $\mathfrak{g}$
- compact group K

on V^K

They are jointly irreducible (no proper subspace invariant for both) iff if G is connected, rep. is irred. for $\mathfrak{g}$.

<u>Generalization</u>: G real reductive (eg. real pts of complex reductive group)
Fix a maximal compact subgroup, algebraic, defined over $\mathbb{R}$ $K \subset G$ (unique upto conjugation in G)

V irred. unitary $\rightsquigarrow V^K = K$-finite vectors
= vectors belonging to finite dim K-invariant subspace

Thm is ~~t~~ still true in this setting.

$SL(2,\mathbb{R})$ example: $\mathfrak{g} = 2\times 2$ trace 0 real matrices

$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = 2\times 2$ trace 0 complex matrices complex reps of real Lie algebra $\mathfrak{g}$ = complex reps of $\mathfrak{g}_{\mathbb{C}}$

$\mathfrak{g}_{\mathbb{C}}$ has basis h, e, f $\quad h = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$e = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad f = \begin{pmatrix} & \\ & \end{pmatrix}$

$$h \cdot (\underbrace{e^{im\theta} |d\theta|^{1/2}}_{v_m}) = m (e^{im\theta} |d\theta|^{1/2})$$

$e \cdot v_m =$ const $\cdot \, v_{m+2}$; $f v_m =$ ~~com~~ const $\cdot \, v_{m-2}$

G real reductive $\supset K$ maximal compact
K acts on $\mathfrak{g} = \mathrm{Lie}(G)$ by automorphisms ("Ad")

$\mathcal{M}(\mathfrak{g}, K)$ = category of $(\mathfrak{g}, K)$-modules

objects: complex vector space X w/ rep. of K, rep. of $\mathfrak{g}$ subject to:

1) Action of ~~$\mathfrak{k}$~~ K locally finite (continuous). any $x \in X$ generates finite dim. space $\langle kx \mid k \in K \rangle$.

By 1) + fin. dim Lie theory: Rep. of K differentiates $\rightsquigarrow$ rep. of $\mathfrak{k} = \mathrm{Lie}(G) \subseteq \mathfrak{g}$.

2) Differential of K action = restriction of $\mathfrak{g}$ action to $\mathfrak{k}$

3) if $Z \in \mathfrak{g}$, $k \in K$, $x \in X$
$k(Zx) = (\mathrm{Ad}(k) Z)(k \cdot x)$

functor: unitary irred. reps $\longrightarrow$ $(\mathfrak{g}, K)$-modules w/ +ve invariant Hermitian form

Going backwards:

<u>Def</u> An <u>invariant Hermitian form</u> on a $(\mathfrak{g}, K)$-module X is $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$, sesquilinear, $\mathbb{C}$-linear in 1st variable, $\mathbb{C}$-conj linear in 2nd variable: $\langle x, x' \rangle = \overline{\langle x', x \rangle}$ preserved by action of K, "differential of preserved" by $\mathfrak{g}$: $\langle Z x, x' \rangle = -\langle x, Z x' \rangle$ for any $Z \in \mathfrak{g} = \mathrm{Lie}(G)$, $x, x' \in X$

<u>Thm</u> (Harish-Chandra)

This functor gives a bijection

$$\left(\begin{array}{c} \text{irred. unitary reps} \\ \text{of } G \text{ (mod unitary} \\ \text{equivalence)} \end{array} \right) \longleftrightarrow \left(\begin{array}{l} \text{irred. } (\mathfrak{g}, K)\text{-mods} \\ \text{w/ +ve invariant} \\ \text{Hermitian form} \end{array} \right)$$

Make $(\mathfrak{g}, K)$-modules more algebraic:

1) Replace real Lie algebra $\mathfrak{g}$ by its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Nothing changes ($\mathfrak{g}_{\mathbb{C}}$ has "complex conjugation" — invariance ~~for $Z \in \mathfrak{g}_{\mathbb{C}}$~~: $\langle Z x, x' \rangle = -\langle x, \bar{Z} x' \rangle$) for $Z \in \mathfrak{g}_{\mathbb{C}}$

Now have reps. of complex reductive $\mathfrak{g}_{\mathbb{C}}$.

<u>Thm</u> K-any compact Lie group.

$C^\infty(K) \supset$ matrix coeffs of finite dim. reps (complex) $\underset{\text{def}}{=} R(K)$

$\uparrow$

fin. gen Hopf-subalgebra of $C^\infty(K)$

~~take the~~
$\rightsquigarrow$ complex alg. group $\text{spec}[R(K)] \underset{\text{def}}{=} K_{\mathbb{C}} \supset K$

Algebraic reps of $K_{\mathbb{C}}$ = locally finite continuous reps of K.

($K_{\mathbb{C}}$ ~~is~~ K may be disconnected if K is complex, reductive, algebraic)
Get all such reps this way.

2) Replace K by $K_{\mathbb{C}}$

$(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ ← complex red. algebraic; acts on $\mathfrak{g}_{\mathbb{C}}$ by $\mathrm{Ad}_{\mathbb{C}}$

↑ complex red. Lie algebra

$\mathcal{M}(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ = complex reps of $\mathfrak{g}_{\mathbb{C}}$ + alg. reps of $K_{\mathbb{C}}$ + compatibility

$\mathcal{M}(\mathfrak{g}, K) = \mathcal{M}(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ ← study using (complex) flag varieties.

<u>Cartan bijection</u>

real forms of $G(\mathbb{C})$ ⟷ alg. auts θ of order 2 of $G(\mathbb{C})$

$\updownarrow$

$G(\mathbb{R})$ real reductive ⟶ $G(\mathbb{C})^{\theta} = K(\mathbb{R})_{\mathbb{C}}$

$\cup$

$K(\mathbb{R})$ maximal compact