

22 Jan 2009

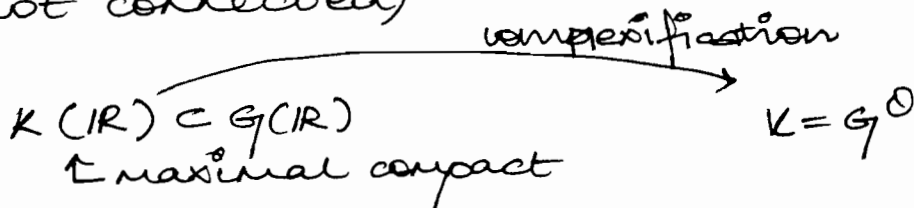
Algebraic Lie Theory at the  
Newton Institute

D. Vogan Representations of Real Reductive  
Lie Groups III

$G$  - complex <sup>connected</sup> reductive algebraic group.  
real form of  $G$  / upto equivalence = automorphism

$\Theta$  of  $G$  of order 1 or 2.

$K = G^\Theta$  - complex reductive group (maybe not connected)



$\mathfrak{g} = \text{Lie}(G)$ ;  $(\mathfrak{g}, K)$ -modules  $\rightsquigarrow$  reasonable reps. of  $G(\mathbb{R})$ .

Study  $(\mathfrak{g}, K)$ -modules via geometry of flag varieties

$B$  = variety of Borel subgroups of  $G$ .  
(projective alg. variety)

Restrict to reps. in which the central  $\mathbb{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$  acts via the trivial character

↑  
poly. algebra in  $\text{rank}(G)$  generators

↙  
is usually called trivial infinitesimal character.

$G$  acts on  $B$  (algebraically)  
 $U(\mathfrak{g}) \rightarrow D(B) \stackrel{\text{def}}{=} \text{algebra of algebraic differential operators on } B.$

Prop (Beilinson - Bernstein)  $\ker U(\mathfrak{g}) \rightarrow D(B)$   
 is the 2-sided ideal generated by  $\ker$  of  $Z(\mathfrak{g})$  action on the trivial rep. (Easy to prove). Furthermore, the map is surjective.

$\mathfrak{g}$ -mod w/ trivial infinitesimal character = modules for  $D(B)$

Can construct these using geometry on  $B$ .

eg:  $Z \subset B$  any closed subset (in the usual topology).

$M = \text{all Schwartz distributions on } B \text{ w/ support in } Z$

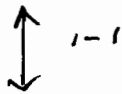
$D(B)$ -module.

WANT:  $\mathfrak{g}$ -mods w/ algebraic action on  $K$ .

Rough idea: use subset  $Z \subset B$  that is  $K$ -stable; take subspace of all distributions on which  $K$  acts algebraically.

can't use delta functions at pts. Need  $\int_{K\text{-orbit}}$  kind of distributions.

Thm (Beilinson-Bernstein)  $\text{Ired. } (\mathfrak{g}, K)$   
- modules w/ trivial infinitesimal character



$\text{Ired. } K$ -equivariant local systems of orbits of  $K$  on  $B$

$\uparrow$   
not too hard, main idea in this story is the surjectivity of  $U(\mathfrak{g}) \rightarrow D(B)$ .

example  $G = SL(2, \mathbb{C})$      $G(\mathbb{R}) = SL(2, \mathbb{R})$   
 $K(\mathbb{R}) = SO(2) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\}$   
 $\theta \in \mathbb{R}$

$$K = SU(2, \mathbb{C}) = \left\{ \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix} \right\}$$

$z \in \mathbb{C}$

$$\mathfrak{Og} = \mathfrak{k} \mathfrak{g}^{-1}$$

$B \cong \text{lines in } \mathbb{P}^2 = \mathbb{CP}^1$

3  $SO(2, \mathbb{C})$  orbits on lines (Witt's theorem).

- isotropic line  $\mathbb{C} \begin{pmatrix} 1 \\ i \end{pmatrix}$
- isotropic line  $\mathbb{C} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$- \text{rest } \cong \mathbb{C}^\times = SO(2, \mathbb{C}) / \pm 1$$

4-  $\text{Ired } (\mathfrak{g}, K)$ -modules w/ trivial character  
 $\mathbb{C} \begin{pmatrix} 1 \\ i \end{pmatrix}$ , trivial local system  $\longleftrightarrow$   $\text{Ired. rep}$   
 $SO(2)$  wts

~~(+1)~~ +2, +4, +6, ...

$\mathbb{C} \begin{pmatrix} 1 & \\ & -i \end{pmatrix}$ , circ. local system  $\Leftrightarrow A \Leftrightarrow$  irred.  
 rema module  $SO(2)$  wts  $-2, -4, -6, \dots$

$\mathbb{C}^*$  has 2 irred. local systems (reps of  $\pm 1/\text{id}$  compact)

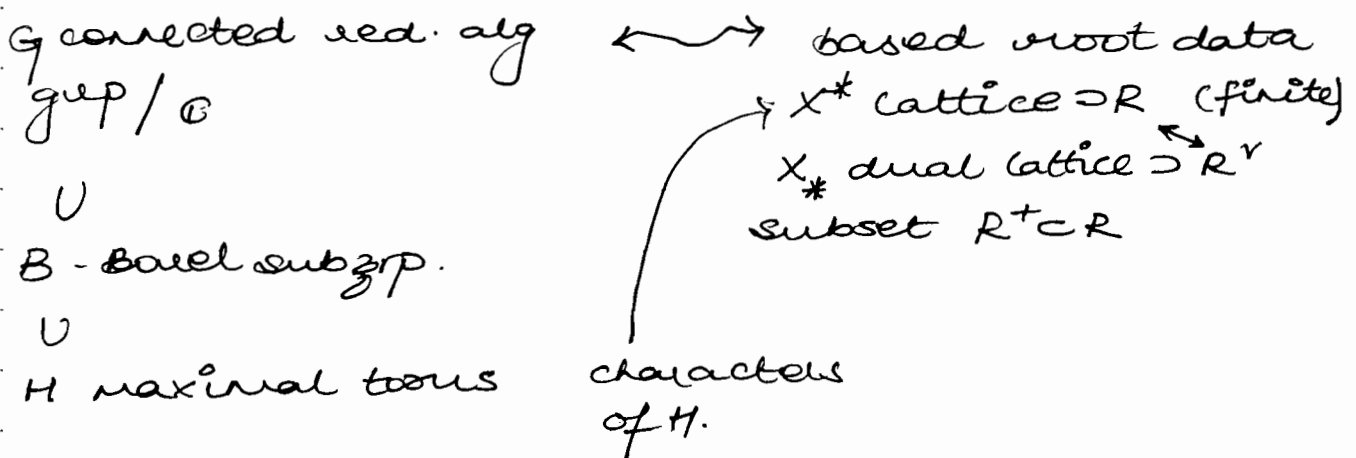
$\mathbb{C} \begin{pmatrix} \# & \\ & \# \end{pmatrix}$   
 trivial local system  $\xleftrightarrow{\text{trivial rep of } SO(2)}$  trivial rep of  $SO(2)$

non trivial local system  $\xleftrightarrow{\text{rep w/ } SO(2)}$  wts  $\pm 1, \pm 3, \pm 5, \dots$

classification of irred.  $(\mathfrak{g}, K)$ -modules  
 is due to Langlands  $\sim 1967$ . + Knapp-Zuckerman et. al.  $\sim 1975$ .

How do we list these pairs (orbit of  $K$  on  $B$ , local system) ?

How do we even list possible  $K$  ?



Goal: describe  $K$ , orbits, local systems etc. in terms of root data.

Looking for autom.  $\theta$  of  $G$  of order 2. By "functoriality" of structure theory means  $\theta \rightsquigarrow$  involute autom.  $z_0$  of based root data. case inf-orientation  $\nearrow$

$$X^* = \mathbb{Z}^n; z_0 = n \times n \text{ matrix of integers } z_0^2 = 1; z_0(R^+) = R^+ \text{ etc.}$$

write  $\Gamma = \{1, z_0\}$  (2-element group) acts on  $G$  preserving  $B, H$ , "pinning"  $\rightsquigarrow G^\Gamma = G \rtimes \Gamma$

Automorphism (algebraic) of  $G$  "inner" to  $z_0 =$  [element of coset  $G \cdot z_0 \subset G^\Gamma$  acting on  $G$  by conjugation].  $Z(G)$ .

$G$ -conjugacy class of "0" =  $G$ -conjugacy class of "almost twisted" involutions. ( $x^2 \in Z(G)$ )

Prop write  $N = N_G(H); H \subset B \subset G$   
 $\downarrow$   
 $z_0$

Every class of almost twisted involutions has a representative  $h \cdot z_0, h \in H$  unique upto conjugacy by  $N$ .

Easy to write conditions on  $h$  to be an almost twisted...

Thm (Adams, ancloux)

(choice of  $x \in \mathfrak{g}^n \setminus \mathfrak{g} \mid x^2 \in Z(\mathfrak{g})$ )  
in  $\mathfrak{g}(\mathbb{B})$  / conjugation by  $G$ ,  $K_x$  orbit  
 $\uparrow$   
stab. of  $x$  in  $\mathfrak{g}$

= (elements  $\lambda z_0 \in N^{\mathbb{B}_0} = N_G(H)^{\mathbb{B}_0}$  s.t.  $(\lambda z_0)^2 \in Z(\mathfrak{g})$ )  
modulo conjugation by  $H$ .

$\uparrow$   
Weyl Tits group calculations.

This gives a computationally effective list of orbits.