22 Jan 2009 Algebraic Lie Theory at the Newton Institute

D. Vogan Representations of Real Reductive Lie Groups III

G - complex connected reductive algebraic group.
real form of G/upto equivalence = automorphism

θ of G of order 1 or 2.

$K = G^\theta$ - complex reductive group (maybe not connected)

$K(\mathbb{R}) \subset G(\mathbb{R}) \xrightarrow{\text{complexification}} K = G^\theta$
↑ maximal compact

$\mathfrak{g} = \mathrm{Lie}(G)$; $(\mathfrak{g}, K)$-modules $\rightsquigarrow$ reasonable reps. of $G(\mathbb{R})$.

study $(\mathfrak{g}, K)$-modules via geometry of flag varieties

$\mathcal{B}$ = variety of Borel subgroups of G.
(projective alg. variety)

Restrict to reps. in which the centre $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ acts via the trivial character

$Z(\mathfrak{g})$: poly. algebra in rank(G) generators

trivial character: usually called trivial infinitesimal character.

G acts on $\mathcal{B}$ (algebraically)

$U(\mathfrak{g}) \longrightarrow \mathcal{D}(\mathcal{B}) \underset{def}{=}$ algebra of algebraic differential operators on $\mathcal{B}$.

Prop (Beilinson - Bernstein) ker $U(\mathfrak{g}) \longrightarrow \mathcal{D}(G)$ is the 2-sided ideal generated by ker of $Z(\mathfrak{g})$ action on the trivial rep. (Easy to prove). Furthermore, the map is surjective.

$\mathfrak{g}$-mod w/ trivial infinitesimal character = modules for $\mathcal{D}(\mathcal{B})$

can construct these using geometry on $\mathcal{B}$.

eg: $Z \subset \mathcal{B}$ any closed subset (in the usual topology).

$M=$ $\underbrace{\text{all Schwartz distributions on } \mathcal{B} \text{ w/ support in } Z}_{\mathcal{D}(\mathcal{B})\text{-module.}}$

WANT: $\mathfrak{g}$-mods w/ algebraic action ~~on~~ of K.

Rough idea: use subset ~~Z~~ $Z \subset \mathcal{B}$ that is K-stable; take subspace of all distributions on which K acts algebraically.

Can't use delta functions at pts. Need $\int_{K\text{orbit}}$ kind of distributions.

<u>Thm</u> (Beilinson - Bernstein) irred. $(\mathfrak{g}, K)$ - modules w/ trivial infinitesimal character

$$\Big\updownarrow {\scriptstyle 1-1}$$

irred. K-equivariant local systems of orbits of K on $\mathcal{B}$

↑
not too hard, main idea in this story is the <u>surjectivity</u> of $U(\mathfrak{g}) \longrightarrow D(\mathcal{B})$.

<u>example</u> $G = SL(2, \mathbb{C}) \quad G(\mathbb{R}) = SL(2, \mathbb{R})$

$$K(\mathbb{R}) = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}_{\theta \in \mathbb{R}}$$

$$K = SO(2, \mathbb{C}) = \left\{ \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix} \right\}_{z \in \mathbb{C}}$$

$$\theta g = {}^t g^{-1}$$

$\mathcal{B} \simeq$ lines in $\mathbb{C}^2 = \mathbb{CP}^1$

3 $SO(2, \mathbb{C})$ orbits on lines (Witt's theorem).

- isotropic line $\mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix}$
- isotropic line $\mathbb{C}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
- rest $\simeq \mathbb{C}^\times = SO(2, \mathbb{C}) / \pm 1$

4 - irred $(\mathfrak{g}, K)$ - modules w/ trivial character

$\mathbb{C}\begin{pmatrix} 1 \\ i \end{pmatrix}$, trivial local system $\longleftrightarrow$ irred. verma

$SO(2)$ wts

$+2, +4, +6, \ldots$

$\mathbb{C}\begin{pmatrix}1\\-i\end{pmatrix}$, triv. local system ~~$\leftrightarrow$~~ $\leftrightsquigarrow$ irred. verma module $SO(2)$ wts $-2, -4, -6, \ldots$.

$\mathbb{C}^\times$ has $\underline{2}$ irred. local systems (reps of ± 1/id component $\simeq \mathbb{Z}/2\mathbb{Z}$)

trivial local system $\leftrightsquigarrow$ trivial rep of $SO(2)$

non trivial local system $\leftrightsquigarrow$ rep w/ $SO(2)$ wts $\pm 1, \pm 3, \pm 5, \ldots$

classification of irred. $(\mathfrak{g}, K)$-modules is due to Langlands ~ 1967. + Knapp-Zuckerman et. al. ~ 1975.

How do we list these pairs (orbit of K on $\mathcal{B}$, local system)?

How do we even list possible K?

G connected red. alg grp / $\mathbb{C}$ $\leftrightsquigarrow$ based root data

X^* lattice $\supset R$ (finite)

X_* dual lattice $\supset R^\vee$

subset $R^+ \subset R$

$\cup$

B - Borel subgrp.

$\cup$

H maximal torus

characters of H.

Goal: describe K, orbits, local systems etc in terms of root data.

Looking for autom. θ of $G_{\mathbb{R}}$ of order 2.
By "functoriality" of structure theory means $\theta \rightsquigarrow$ involute autom. $\mathcal{z}_0$
lose inf-ormation $\longrightarrow$ of based root data.

$X^* = \mathbb{Z}^n$; $\mathcal{z}_0 = n \times n$ matrix of integers $\mathcal{z}_0^2 = 1$; $\mathcal{z}_0(R^+) = R^+$ etc.

write $\Gamma = \{1, \mathcal{z}_0\}$ (2-element group) acts on G preserving B, H, "pinning"
$\rightsquigarrow G^\Gamma \underset{def}{=} G \rtimes \Gamma$

Automorphism (algebraic) of G "inner" to $\mathcal{z}_0 \underset{Thm}{=}$ [element of coset $G \cdot \mathcal{z}_0 \subset G^\Gamma$ acti-ing on G by conjugation]$\cdot Z(G)$.

G-conjugacy class of "θ"
$=$ G-conjugacy class of "almost twisted" involutions. ($x^2 \in Z(G)$)

<u>Prop</u> Write $N = N_G(H)$; $H \subset B \subset G \circlearrowleft \mathcal{z}_0$

Every class of almost twisted involutions has a representative $h \cdot \mathcal{z}_0$, $h \in H$ unique upto conjugacy by N.

Easy to write conditions on h to be an almost twisted ...

Thm (Adams, du Cloux)

(choice of $x \in G^\Gamma \setminus G \mid x^2 \in Z(G)\}$, K_x orbit in B) / conjugation by G

stab. of x in G

$= ($elements $n\,\delta_0 \in N^{\delta_0} = N_G(H)^{\delta_0}$ s.t. $(n\delta_0)^2 \in Z(G))$ modulo conjugation by H.

~~weyl~~ Tits group calculations.

This gives a computationally effective list of orbits.