23 Jan 2009 — Algebraic Lie Theory at the Newton Institute

D. Vogan — Representations of Real Reductive Lie Groups IV

How to find unitary representations among irred. $(\mathfrak{g}, K)$-modules.

G complex algebraic
$\cup$
B_0
$\cup$
T_0
} pinning

δ_0 aut. of order 2 preserving pinning
$\Gamma = \{1, \delta_0\}$
$G^\Gamma = G \rtimes \Gamma$

fix $x \in G^\Gamma \setminus G$, $x^2 \in Z(G)$

$K =_{def} G^x$ (complexification of maximal compact of a real form of $G(\mathbb{R})$)

Beilinson–Bernstein:

$$\left\{\begin{matrix}\text{irred. } (\mathfrak{g},K)\text{-mods} \\ \text{w/ trivial infinit-} \\ \text{esimal character}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{irred. } K\text{-eqvt local} \\ \text{systems on orbits of} \\ K \text{ on } \mathcal{B} = \text{flag variety}\end{matrix}\right\}$$

(In category $\mathcal{O}$ any irred $L(\lambda)$ is canonical quotient of the Verma $M(\lambda)$.
$K_0(\mathcal{O})$ free $\mathbb{Z}$-module w/ basis $\{[L(\lambda)]\}$
OR " " " $\{[M(\lambda)]\}$

KL-theory: find change of basis)

"λ" = irred. K-equivariant local system on orbits of K on $\mathcal{B}$.

$L(\lambda)$ = corresponding irred. $(\mathfrak{g}, K)$-module.

= canonical quotient of "standard" $(\mathfrak{g}, K)$-module $M(\lambda)$.

Same story as category $\mathcal{O}$ holds here.

$P_{\mu,\lambda}$ = poly. in q, coeffs $\longleftrightarrow$ multiplicity of local systems μ in H^* sheaves of perverse extensions of λ.

$$L(\lambda) = \sum_{\mu} (-1)^{|\mu| - |\lambda|} P_{\mu,\lambda}(1)\, M(\mu).$$

$|\mu|$ = dim of K-orbit under μ.

Kazhdan-Lusztig algorithm to compute $P_{\mu,\lambda}$ based on:

For each simple root s, $\mathbb{P}^1$-fibration $\mathcal{B} \xrightarrow{\pi_s} \mathcal{P}_s$ use decomposition theorem for maps π_s.

<u>Finding unitary reps</u>: Have $(\mathfrak{g}, K)$-module V. Want to know if V admits an invariant Hermitian form and whether this form is definite.

1st part answered by Knapp-Zuckerman (1970s):

can define new $(\mathfrak{g}, K)$-module V^h = K-finite linear maps $V \to \mathbb{C}$ endowed w/ opposite complex structure.

Easy: V^h is another irred $(\mathfrak{g}, K)$-module.
Easy: V admits an non-degenerate invariant Hermitian form $\Longleftrightarrow V^h \underset{(\mathfrak{g},K)\text{-mod}}{\simeq} V$

An invariant sesquilinear pairing on V = $(\mathfrak{g}, K)$-morphism $V \longrightarrow V^h$

Issue: What's the map $V \to V^h$ in Beilinson - Bernstein picture?

x acts on the category of $(\mathfrak{g}, K)$-mods
twist action of $\mathfrak{g}$ by x, dont change action of K. ~~V=~~
$V \longrightarrow V^x$

Prop (Adams) $V^x \simeq V^h$
proof $V \to V^h$ computed by Knapp-Zuckerman.

Defn $K^\Gamma = \langle K, x \rangle$; $1 \to K \to K^\Gamma \to \Gamma \to 1$

Clifford theory: irred. rep of K^Γ = $\left\{ \begin{array}{l} \text{extension an irred rep of } K \text{ fixed by } x \text{ — two possibilities} \\ \text{OR} \\ \mathrm{Ind}_K^{K^\Gamma} (\text{irred of } K \text{ not fixed by } x) \end{array} \right.$

Theory of $(\mathfrak{g}, K)$-mods w/ invari-ant Hermitian forms
$\updownarrow$
K^Γ-equivariant perverse sheaves on $\mathcal{B}$.

irred $(\mathfrak{g}, K)$-mod w/ trivial character + invariant form $\longleftrightarrow$ K-eqvt. local systems that extend to K^Γ-eqv. (in 2-ways.)

always 2 equiv classes of form +/-

Extend KL-algorithm to compute perverse extensions of K^Γ-eqvt. local systems ... (ROUTINE).

<u>Conclusion</u>: can (explicitly) write invariant Herm. form on an irred. $(\mathfrak{g}, K)$-module (if it exists) as an integer combination of invariant forms on standard modules
non-degen.

OOPS - Verma modules - radical of Shapovalov form has to be big.

Back to Verma modules $M_1(\lambda) \longrightarrow M(\lambda) \twoheadrightarrow L(\lambda)$.
Shapovalov: put an inv. form:

$\langle \cdot , \cdot \rangle_\lambda$ on $M(\lambda) \simeq U(\mathfrak{n}^-)$

family of forms on $U(\mathfrak{n}^-)$; $U(\mathfrak{n}^-)_\lambda \perp U(\mathfrak{n}^-)_\mu$, $\lambda \neq \mu$. So a form on each wt. space $U(\mathfrak{n}^-)_\mu$ ~~which are~~ the matrix of which depends polynomially on λ. Form on 0 wt space $= 1$.

$M_1(\lambda) =$ radical of $\langle \cdot , \cdot \rangle_\lambda$. So if $L(\lambda) \neq M(\lambda)$, then form is degenerate.

<u>Jantzen</u>: defined a natural filtration of $M(\lambda)$ (order of vanishing of $\langle \cdot , \cdot \rangle_\lambda$).

$M(\lambda) = M_0(\lambda) \supset M_1(\lambda) \supset M_2(\lambda) \supset \cdots \supset M_n(\lambda) = 0$
Showed: Taking residues $\rightsquigarrow$ natural <u>non degenerate</u> $\langle \cdot , \cdot \rangle^{(i)}$ on $M_i(\lambda) / M_{i+1}(\lambda)$

in $K_0(\mathcal{O})$

$$M(\lambda) \rightsquigarrow gr\, M(\lambda) = \bigoplus_{i=0}^{n-1} \underbrace{M^i(\lambda)/M^{i+1}(\lambda)}_{\substack{\text{endowed w/} \\ \langle\cdot,\cdot\rangle^{(i)} \text{ (nondegenerate)}}}$$

Harish-Chandra modules

(Key to Jantzen construction is a deformation of λ.)

Turns out that any standard HC-module $M(\lambda)$ appears canonically in a continuous family; continuous parameter $\nu \in \mathfrak{a}^* \leftarrow -1$ eigenspace of Cartan involution on some θ-stable Cartan subalgebra

$\{\lambda = (\lambda_{dis}, \nu(\lambda))$

$\rightsquigarrow$ 1-param. family $M(\lambda_{dis}, t\nu(\lambda))$

$\rightsquigarrow$ Jantzen filtration $\rightsquigarrow$ non-degenerate form $\langle\cdot,\cdot\rangle$ on $gr\, M(\lambda_{dis}, t\nu)$

More or less: defines non-degen. form on each standard module (in K_0).

- writes form on irred. as computable combination of forms on standards.

UNITARY? These forms on standards are "peculiar":

$SL(2,\mathbb{R})$: spherical principal series: sum of all even chars of $SO(2)$, parameter $t \geq 0$.

signature?

-6	-4	-2	0	2	4	6	8	
+	+	+	+	+	+	+	+	$0 \le t < 1$
−	−	−	+	−	−	−	−	$1 \le t < 3$
+	+	−	+	−	+	+	+	$3 \le t < 5$

To detect unitary reps., need to compute the signatures for standard reps.

<u>Thm</u> (Jantzen) signature of $\langle \cdot, \cdot \rangle_t$ is semi-continuous in t. change in signature for from $t-\varepsilon$ to t = sum of signatures on odd levels of Jantzen filtration for $\langle \cdot, \cdot \rangle_t$

↳ computed by KL-theory.