Final Exam Solutions

Problem A [1994]: *Suppose the coefficients of the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n \, z^n$$

satisfy the recurrence relation

$$a_0 = 1, \qquad a_1 = -1, \qquad 3a_n + 4a_{n-1} - a_{n-2} = 0 \text{ for } n \geq 2.$$

Find the radius of convergence and the function to which it converges.

Let us first show that this series *does* have a appositive radius of convergence R, so that it does converge to some analytic function on an open neighborhood of 0. To do this, it suffices to show that $|a_n| \leq 2^n$ for all n, since this implies

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} \leq \overline{\lim_{n\to\infty}} \sqrt[n]{2^n} = 2,$$

which would imply by Hadamard's Theorem that $R > \frac{1}{2}$. The inequality follows by a quick induction, for $|a_0| = 2^0$ and $|a_1| < 2^1$, and (assuming the inductive hypothesis holds)

$$\begin{aligned} |a_n| &= \left| \frac{a_{n-2} - 4a_{n-1}}{3} \right| \leq \frac{1}{3}\big(|a_{n-2}| + 4|a_{n-1}|\big) \leq \frac{1}{3}\big(2^{n-2} + 4 \cdot 2^{n-1}\big) \\ &= 2^{n-2}\,\frac{1+8}{3} = 3 \cdot 2^{n-2} < 4 \cdot 2^{n-2} = 2^n. \end{aligned}$$

Thus, the series converges with some positive radius R. To determine R precisely, let us first find what the series sums to. Since the convergence is absolute, we can multiply and rearrange our series to obtain:

$$f(z) = \sum_{n=0}^{\infty} a_n \, z^n, \quad z\, f(z) = \sum_{n=1}^{\infty} a_{n-1} \, z^n, \qquad z^2 \, f(z) = \sum_{n=2}^{\infty} a_{n-2} \, z^n.$$

Thus,

$$\begin{aligned} 3\, f(z) + 4z\, f(z) - z^2\, f(z) &= 3(a_0 + a_1\, z) + 4(a_0\, z) + \sum_{n=2}^{\infty} \big(2a_n + 4a_{n-1} - a_{n-2}\big) z^n \\ &= 3(a_0) + (3a_1 + 4a_0) z + \sum_{n=2}^{\infty} 0. \end{aligned}$$

Consequently, we find

$$(3 + 4z - z^2) f(z) = 3(1) + (-3 + 4) z \quad \Longrightarrow \quad f(z) = \frac{3 + z}{3 + 4z - z^2}.$$

Returning to the radius of convergence, it is the minimum distance from the center of the series 0 to the nearest singularity of f. Since the denominator of f vanishes at $z = 2 + \sqrt{7}$ and $z = 2 - \sqrt{7}$, we conclude

$$R = \min\big\{|2 + \sqrt{7}|, |2 - \sqrt{7}|\big\} = \sqrt{7} - 2.$$

(Observe that $R \approx 0.65$, which is indeed greater than the initial estimate of $1/2$.)

Problem B [1977]: *Let f be an analytic function whose Taylor series for $|z| < 1$ is $1 + 2z + 3z^2 + \cdots$. Define the sequence of real numbers $a_0, a_1, a_2, \ldots$ by the formula*

$$f(z) = \sum_{n=0}^{\infty} a_n (z+2)^n.$$

Calculate the a_n and determine the radius of convergence of

$$g(z) = \sum_{n=0}^{\infty} a_n\, z^n?$$

Consider the geometric series

$$g(z) = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z},$$

which converges absolutely for $|z| < 1$. Since term-by-term differentiation is valid inside the radius of convergence, we find

$$g'(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

which agrees with our initial series, whence

$$f(z) = g'(z) = \frac{d}{dz}\left\{\frac{1}{1-z}\right\} = \frac{1}{(1-z)^2}.$$

Next, observe that the a_n are merely the coefficients of the Taylor series expansion of $f(z)$ at the point $z = -2$. Note that (by induction)

$$f'(z) = \frac{2}{(1-z)^3}, \quad f''(z) = \frac{3 \cdot 2}{(1-z)^4}, \quad \ldots, \quad f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}},$$

we calculate that

$$a_n = \frac{1}{n!} f^{(n)}(-2) = \frac{1}{n!}\,\frac{(n+1)!}{3^{n+2}} = \frac{n+1}{3^{n+2}}.$$

Lastly, consider the power series $g(z) = \sum a_n\, z^n$. Since $f(z) = \sum a_n (z+2)^n$, we conclude by the uniqueness of analytic functions that

$$g(z) = \sum_{n=0}^{\infty} a_n\, z^n = f(z-2) = \frac{1}{(3-z)^2}.$$

Hence, the only singularity of g occurs at $z = 3$, whence the radius of convergence for g is 3.

Alternatively, observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{3^{n+3}} \cdot \frac{3^{n+2}}{n+1} = \frac{1}{3}\,\frac{n+2}{n+1} \to \frac{1}{3} \quad \text{as} \quad n \to \infty,$$

we conclude again that $R = 1/(\frac{1}{3}) = 3$.

Problem C [1990]: *Suppose f and g are analytic in the entire complex plane, and suppose that $\left|f(z)\right| \leq \left|g(z)\right|$ for every $z \in \mathbb{C}$. Prove that there is a complex constant A such that $f(z) = A\,g(z)$ for all z.*

Notice that if $g(z) \equiv 0$, then the inequality implies $f(z) \equiv 0$ as well, and the result holds for any choice of A. Hence, let us assume that g is not identically zero.

Define the function

$$h(z) = \frac{f(z)}{g(z)}.$$

Observe that h is analytic on $\mathbb{C}$ except for isolated singularities, namely, the (isolated) zeros of $g(z)$. Moreover, if z is not a singularity, we have

$$\left|h(z)\right| = \frac{|f(z)|}{|g(z)|} \leq 1. \tag{1}$$

We claim that these are removable singularities for $h(z)$. To see this, let z_0 be a singularity of h, i.e. a zero of g. As this zero is isolated, there exists a deleted neighborhood $B = A(z_0; 0, R)$ of z_0 on which g is nonzero. But then equation (1) holds for every $z \in B$. This implies that z_0 is neither a pole nor essential, since either of those conditions implies f is unbounded in any neighborhood of z_0. Hence, z_0 must be removable.

Consequently, h can be made continuous and analytic at each removable singularity z_0, and so h is thus analytic on the whole of $\mathbb{C}$. Moreover, since equation (refeqn:1) holds for every non-singularity, continuity implies (refeqn:1) holds at each singularity too.

Thus, h is everywhere bounded by 1. But Liouville's Theorem asserts that h must therefore be a constant function, say $h(z) \equiv A$. This implies that

$$f(z) \equiv A\,g(z),$$

where equality holds for all points at which $g(z) \neq 0$. But since $g(z) = 0$ implies $f(z) = 0$, we have equality at all points of $\mathbb{C}$.

Problem D [1978]: *Let $f(z) = u(z) + i\,v(z)$ be analytic in $\{|z| < 1\}$. If $f(0) = 0$, then show that*

$$\int_0^{2\pi} u(r\,e^{i\theta})^2\,d\theta = \int_0^{2\pi} v(r\,e^{i\theta})^2\,d\theta$$

for any $0 < r < 1$.

Since f is analytic, so too is f^2. Thus, according to the Cauchy Integral Formula,

$$0 = f(0)^2 = \frac{1}{2\pi\,i} \int_{|z|=r} \frac{f(z)^2}{z-0}\,dz$$

for any $0 < r < 1$.

Using the parametrization $z(\theta) = r\,e^{i\theta}$ for $0 \le \theta \le 2\pi$, we find

$$0 = \frac{1}{2\pi\,i} \int_0^{2\pi} \frac{f(r\,e^{i\theta})^2}{r\,e^{i\theta}} r\,i\,e^{i\theta}\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(r\,e^{i\theta})^2\,d\theta.$$

(Astute eyes will recognize this as just the Mean Value Property applied to the analytic function f^2.) Multiplying through by 2π yields

$$0 = \int_0^{2\pi} f(r\,e^{i\theta})^2\,d\theta.$$

Now, if u and v denote the real and imaginary parts of f, then

$$f^2 = (u + i\,v)(u + i\,v) = (u^2 - v^2) + 2uv\,i,$$

whence

$$\begin{aligned} 0 &= \int_0^{2\pi} \left(u(r\,e^{i\theta})^2 - v(r\,e^{i\theta})^2\right) + 2u(r\,e^{i\theta})v(r\,e^{i\theta})\,i\,d\theta \\ &= \int_0^{2\pi} u(r\,e^{i\theta})^2 - v(r\,e^{i\theta})^2 d\theta + 2i \int_0^{2\pi} u(r\,e^{i\theta})v(r\,e^{i\theta})\,d\theta. \end{aligned}$$

Since u and v are real-valued, the two integrals above are *real numbers*. Thus, equating real parts yields

$$0 = \int_0^{2\pi} u(r\,e^{i\theta})^2 - v(r\,e^{i\theta})^2 d\theta = \int_0^{2\pi} u(r\,e^{i\theta})^2\,d\theta - \int_0^{2\pi} v(r\,e^{i\theta})^2 d\theta,$$

which completes the proof.

As a bonus, equating the imaginary parts gives us another cool result, namely

$$\int_0^{2\pi} u(r\,e^{i\theta})\,v(r\,e^{i\theta})\,d\theta = 0.$$

Problem E [1979]: *For what $z \in \mathbb{C}$ does the series below converge? To what function does the series converge?*

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n}{z^n} \right)$$

Observe this is a Laurent series: the sum of a power series and a singular series. Hence, this series will converge for $z \in \mathbb{C}$ if and only if both the power series (regular part) converges at z and the singular series converges at z.

Observe that the regular part is

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

This is precisely the Taylor series of the complex exponential e^z. As this is an entire function with no singularities, its radius of convergence is infinite. Alternatively, note that

$$\left| \frac{c_{n+1}}{c_n} \right| = \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \frac{1}{n+1} \to 0 \quad \text{as} \quad n \to \infty,$$

whence $R = 1/0 = \infty$.

On the other hand, the singular part is

$$\sum_{n=1}^{\infty} \frac{n}{z^n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots .$$

Recall from Problem B that

$$\sum_{n=0}^{\infty} (n+1) z^n = 1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2},$$

and this converges for $|z| < 1$. Hence, if $|z| > 1$, then

$$\frac{1}{z} \cdot \frac{1}{\left(1 - \frac{1}{z}\right)^2} = \frac{1}{z} + \frac{2}{z^2} + \frac{1}{z^3} + \cdots ,$$

whence the singular part sums to

$$\frac{1}{z} \cdot \frac{1}{\left(1 - \frac{1}{z}\right)^2} = \frac{z}{(1-z)^2}$$

and converges for $|z| > 1$.

Thus, the Laurent series sums to

$$e^z + \frac{z}{(1-z)^2}$$

and converges on the annulus for $|z| < 1$.

Problem F [1978]: *Show there is a complex analytic function defined on the set* $U = \{|z| > 4\}$ *whose derivative is*

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there an analytic function on U *whose derivative is*

$$\frac{z^2}{(z-1)(z-2)(z-3)}?$$

Let us set $f(z) = z(z-1)^{-1}(z-2)^{-2}(z-3)^{-1}$. Since f is analytic in the annulus $|z| > 4$, it has a Laurent expansion there. Using partial fractions, we find that

$$f(z) = \frac{1}{2}\frac{1}{z-1} - 2\frac{1}{z-2} + \frac{3}{2}\frac{1}{z-3}.$$

Now, if $|z| > 4$, then $|\frac{k}{z}| < \frac{k}{4} < 1$ for any $0 < k < 4$, whence

$$\frac{1}{z-k} = \frac{1}{z}\frac{1}{1-\frac{k}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{k}{z}\right)^n = \sum_{n=1}^{\infty} k^{n-1}\frac{1}{z^n}.$$

Hence, f has the Laurent expansion

$$f(z) = \sum_{n=1}^{\infty}\left(\frac{1}{2}(1^{n-1}) - 2(2^{n-1}) + \frac{3}{2}3^{n-1}\right)\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1-2^{n+1}+3^n}{2}\frac{1}{z^n}$$
$$= \frac{1}{z^2} + \frac{6}{z^3} + \frac{25}{z^4} + \frac{90}{z^5} + \cdots$$

Notice that the coefficient of the first term vanished, so this series can just as well start at $n = 2$. Moreover, the resulting Laurent series actually converges on the larger annulus $|z| > 3$.

If we define F by the Laurent series

$$F(z) := \sum_{n=2}^{\infty}\frac{1-2^{n+1}+3^n}{2(1-n)}\frac{1}{z^{n-1}} = \sum_{n=1}^{\infty}\frac{1-2^{n+2}+3^{n+1}}{-2n}\frac{1}{z^n},$$

then F converges on the same annulus, and term-by-term differentiation shows that $F'(z) = f(z)$, so F is an analytic antiderivative of f on the set $|z| > 4$.

Now, set $g(z) = z^2(z-1)^{-1}(z-2)^{-2}(z-3)^{-1}$; we shall show g has no analytic antiderivative on the annulus $|z| > 4$. To see this, note that *if* such an antiderivative existed, then the Fundamental Theorem of Calculus implies that $\int_\gamma g\,dz = 0$ for any closed curve γ in the annulus. However, since $g(z) = z\,f(z)$, we conclude

$$g(z) = \sum_{n=1}^{\infty}\frac{1-2^{n+1}+3^n}{2}\frac{1}{z^{n-1}} = \frac{1}{z} + \frac{6}{z^2} + \frac{25}{z^3} + \frac{90}{z^4} + \cdots$$

whence term-by-term integration and Cauchy's Integral Formula imply

$$\int_{|z|=5} g(z)\,dz = \int_{|z|=5}\frac{1}{z}\,dz + \int_{|z|=5}\frac{6}{z^2}\,dz + \int_{|z|=5}\frac{25}{z^3}\,dz + \cdots = 2\pi i + 0 + 0 + \cdots = 2\pi i \neq 0.$$

Thus, g has no antiderivative on the annulus.

Problem G [1980]: *Let n be a positive integer. Let C_n denote the circle $\{|z| = n\}$, oriented positively. Use residues to find all possible values of*

$$\int_{C_n} \frac{z + e^z}{z^2(2z-5)(3z-10)}\, dz.$$

Let f denote the integrand, and observe that

$$f(z) = \frac{z + e^z}{6\, z^2(z - \frac{5}{2})(z - \frac{10}{3})},$$

so f has isolated singularities (poles) at 0, $\frac{5}{2}$, and $\frac{10}{3}$. Let us calculate the residues at each singularity.

For the singularity $\frac{5}{2}$, observe that we can write f in the form

$$f(z) = \frac{1}{z - \frac{5}{2}} \left(\frac{z + e^z}{6\, z^2(z - \frac{10}{3})} \right),$$

where the expression in parentheses is analytic on a neighborhood of $\frac{5}{2}$, and hence has a *power series* expansion there of the form $c_0 + c_1(z - \frac{5}{2}) + c_2(z - \frac{5}{2})^2 + \cdots$. Hence,

$$\operatorname{Res}\left(f, \frac{5}{2}\right) = c_0 = \left.\frac{z + e^z}{6\, z^2(z - \frac{10}{3})}\right|_{z=5/2} = -\frac{4}{125}\left(\frac{5}{2} + e^{5/2}\right) =: B.$$

Similar reasoning leads to

$$\operatorname{Res}\left(f, \frac{10}{3}\right) = \left.\frac{z + e^z}{6\, z^2(z - \frac{5}{2})}\right|_{z=10/3} = \frac{9}{500}\left(\frac{10}{3} + e^{10/3}\right) =: C.$$

For the last singularity 0, we modify our approach slightly. Observe that we can write f in the form

$$f(z) = \frac{1}{z^2} \left(\frac{z + e^z}{6(z - \frac{5}{2})(z - \frac{10}{3})} \right),$$

where the expression in parentheses is analytic on a neighborhood of 0, and hence has a *power series* expansion there of the form $b_0 + b_1\, z + b_2\, z^2 + \cdots$. Hence,

$$\operatorname{Res}(f, 0) = b_1 = \left.\frac{d}{dz}\left\{ \frac{z + e^z}{6(z - \frac{5}{2})(z - \frac{10}{3})} \right\}\right|_{z=0} = \frac{27}{500} =: A.$$

Now, since $0 < 2 < \frac{5}{2} < 3 < \frac{10}{3} < 4$, the Residue Theorem implies

$$\int_{C_n} \frac{z + e^z}{z^2(2z-5)(3z-10)}\, dz = \begin{cases} 2\pi\, i\, A & n = 1, 2 \\ 2\pi\, i(A + B) & n = 3 \\ 2\pi\, i(A + B + C) & n = 4, 5, \ldots \end{cases}$$

Problem H [1986]: *Let γ be a Jordan curve enclosing the points $0, 1, 2, \ldots, k$. Use residues to evaluate the integrals*

$$I_k := \int_\gamma \frac{dz}{z(z-1)(z-2)\cdots(z-k)}, \qquad k = 0, 1, 2, \ldots$$
$$J_k := \int_\gamma \frac{(z-1)(z-2)\cdots(z-k)}{z}\, dz, \qquad k = 0, 1, 2, \ldots$$

Let us start with the J_k first. In this case, the integrand $f_k(z)$ has only one singularity, namely 0, and so

$$\mathrm{Res}(f_k, 0) = (z-1)(z-2)\cdots(z-k)\big|_{z=0} = (-1)(-2)\cdots(-k) = (-1)^k\, k!.$$

Observe that this also makes sense if $k = 0$, since the residue of $frac1z$ at 0 is $1 = 0!$. Hence, by the Residue Theorem

$$J_k = 2\pi(-1)^k k!\, i.$$

As for the I_k, observe that by Cauchy's Integral Formula,

$$I_0 = \int_\gamma \frac{1}{z}\, dz = 2\pi\, i.$$

So let us assume $k \geq 1$. In this case, the integrand g_k has $k+1$ zeros, namely, the integers $0, 1, 2, \ldots, k$. Since each of these are simple poles, arguing as in Problem G implies that for $0 \leq j \leq k$,

$$\begin{aligned}\mathrm{Res}(g_k, j) &= \frac{1}{z(z-1)(z-2)\cdots(z-j+1)(z-j-1)\cdots(z-k)}\Bigg|_{z=j} \\ &= \frac{1}{j(j-1)(j-2)\cdots(2)(1)(-1)(-2)\cdots\big(-(k-j)\big)} = \frac{(-1)^{k-j}}{j!(k-j)!}\end{aligned}$$

Thus, by the Residue Theorem, it follows that

$$I_k = 2\pi\, i \sum_{j=0}^{k} \mathrm{Res}(g_k, j) = 2\pi\, i \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!}.$$

However, this can be simplified greatly by recalling that the *binomial coefficients* are given by

$$\binom{n}{r} := \frac{n!}{r!(n-r)!},$$

whence

$$I_k = \frac{2\pi\, i}{k!} \sum_{j=0}^{k} \binom{k}{j} 1^j (-1)^{k-j} = \frac{2\pi\, i}{k!}\big(1 + (-1)\big)^k = 0.$$

Problem I [1997]: *Use residues to evaluate:*

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\, dx.$$

Observe that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{i\,x}}{1+x+x^2}\, dx,$$

so we shall evaluate the latter integral and take its real part.

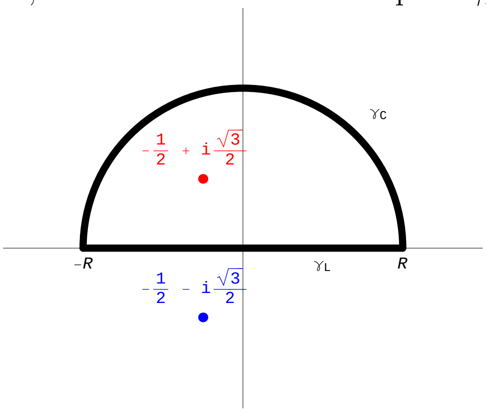
Observe that the integrand can be written as

$$f(z) := \frac{e^{i\,z}}{1+z+z^2} = \frac{e^{iz}}{\left(z-(\frac{-1}{2}+i\frac{\sqrt{3}}{2})\right)\left(z-(\frac{-1}{2}-i\frac{\sqrt{3}}{2})\right)}.$$

Since the two singularities of f are simple, we find

$$\mathrm{Res}\left(f, \frac{-1}{2}+i\frac{\sqrt{3}}{2}\right) = \left.\frac{e^{iz}}{z-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)}\right|_{z=\frac{-1}{2}+i\frac{\sqrt{3}}{2}} = \frac{e^{-\sqrt{3}/2-i/2}}{i\sqrt{3}} = -\frac{i\,e^{-i/2}}{\sqrt{3}\,e^{\sqrt{3}/2}}.$$

Now, for any radius $R > 1$, consider the semicircular path γ_R below:



In this case, γ_L is the line segment from $-R$ to R and γ_C is the circular arc centered at 0 of radius R.

By the Residue Theorem, we have

$$\int_{\gamma_L} f(z)\, dz + \int_{\gamma_C} f(z)\, dz = \int_{\gamma_R} f(z)\, dz = 2\pi\, i\, \mathrm{Res}\left(f, \frac{-1}{2}+i\frac{\sqrt{3}}{2}\right) = \frac{2\pi\, e^{-i/2}}{\sqrt{3}\, e^{\sqrt{3}/2}},$$

and this holds for every $R > 1$.

Now, under the parametrization $z(x) = x$ for $-R \leq x \leq R$, we have

$$\int_{\gamma_L} f(z)\, dz = \int_{-R}^{R} \frac{e^{ix}}{1+x+x^2}\, dx.$$

On the other hand, observe that if $z \in \gamma_C$, then

$$\big|f(z)\big| = \frac{|e^{iz}|}{\big|z - (\frac{-1}{2} + i\frac{\sqrt{3}}{2})\big|\,\big|z - (\frac{-1}{2} - i\frac{\sqrt{3}}{2})\big|} \le \frac{e^{\mathrm{Re}(iz)}}{\big||z| - |\frac{-1}{2} + i\frac{\sqrt{3}}{2}|\big|\,\big||z| - |\frac{-1}{2} - i\frac{\sqrt{3}}{2}|\big|}$$
$$= \frac{e^{-\operatorname{Im} z}}{(R-1)(R-1)} \le \frac{1}{(R-1)^2}.$$

Thus,

$$\left|\frac{2\pi\, e^{-i/2}}{\sqrt{3}\, e^{\sqrt{3}/2}} - \int_{-R}^{R} \frac{e^{ix}}{1+x+x^2}\,dx\right| = \left|\int_{\gamma_R} f(z)\,dz - \int_{\gamma_L} f(z)\,dz\right| \le \left|\int_{\gamma_C} f(z)\,dz\right|$$
$$\le \int_{\gamma_C} |f(z)|\,|dz| \le \frac{1}{(R-1)^2}\cdot \pi\, R = \frac{\pi R}{(R-1)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x+x^2}\,dx = \frac{2\pi\, e^{-i/2}}{\sqrt{3}\, e^{\sqrt{3}/2}}.$$

Consequently

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x+x^2}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{i\,x}}{1+x+x^2}\,dx = \mathrm{Re}\left(\frac{2\pi\, e^{-i/2}}{\sqrt{3}\, e^{\sqrt{3}/2}}\right)$$
$$= \frac{2\pi}{\sqrt{3}\, e^{\sqrt{3}/2}}\,\mathrm{Re}(e^{-i/2}) = \frac{2\pi}{\sqrt{3}\, e^{\sqrt{3}/2}}\cos\left(-\frac{1}{2}\right) = \frac{2\pi\,\cos(1/2)}{\sqrt{3}\, e^{\sqrt{3}/2}}.$$