Midterm Solutions, Part 1

Problem 1. *Consider the complex-valued functions*

$$f(z) = x^3 + i\,y^3, \qquad g(z) = \frac{x - i\,y}{x^2 + y^2}.$$

(a) Give the definition of complex differentiability for complex functions. How do the Cauchy-Riemann equations relate to complex differentiability?

A complex function $f : U \to \mathbb{C}$ is *complex differentiable* at z_0 if the (complex) limit

$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and is finite.

The *Cauchy-Riemann equations* for a complex function $f = u + i\,v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

It relates to complex differentiability via the following theorem: a complex function f is complex differentiable at z_0 if and only if it is real differentiable at z_0 and satisfies the Cauchy-Riemann equations there.

(b) Show that f is differentiable at $z = 1 + i$. At what other points is f is $\mathbb{C}$-differentiable?

Taking partial derivatives with $u(x, y) = x^3$ and $v(x, y) = y^3$, we find

$$\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 3y^2.$$

Thus, f is C^1. Moreover, the CR equations hold provided $3x^2 = 3y^2$, or $x^2 = y^2$. This means f is holomorphic on the lines $y = \pm x$. In particular, f is holomorphic at $z = 1 + i$, since it's on the line $y = x$.

(c) Show that g is differentiable at $z = 1 + i$. At what other points is g is $\mathbb{C}$-differentiable?

Observe that

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x - i\,y}{x^2 + y^2} = g(z).$$

Thus, g is the quotient of two polynomials (which are holomorphic on $\mathbb{C}$), and so g is differentiable whenever the denominator is nonzero. Thus, g is differentiable for all $z \neq 0$, and in particular at $z = 1 + i$.

(d) Calculate $f'(1 + i)$ and $g'(1 + i)$. What are the amplitwist effects of these derivatives, i.e. through what angles are tangents rotated, and by what amplification factor?

Recall that if f is differentiable, then $f' = u_x + i\,v_x$. Thus,

$$f'(1 + i) = \frac{\partial u}{\partial x}(1 + i) + i\,\frac{\partial v}{\partial x}(1 + i) = 3(1)^2 + i\,0 = 3.$$

Since $3 = 3\,e^{i0}$ in polar form, the infinitesimal amplitwist of f at $1 + i$ is a dilation by a factor of 3 with no rotation.

On the other hand, by the quotient rule,

$$g'(z) = -\frac{1}{z^2} \quad \Longrightarrow \quad g'(1 + i) = -\frac{1}{(1 + i)^2} = -\frac{1}{2i} = \frac{i}{2}.$$

Since $i/2 = \frac{1}{2}\,e^{i\,\pi/2}$ in polar form, the infinitesimal amplitwist of g at $1 + i$ is a contraction by a factor of $1/2$ and a rotation 90° counterclockwise.

Problem 2. *(a) State Cauchy's Integral Theorem.*

Suppose $U \subset \mathbb{C}$ is a simply connected open set and $f : U \to \mathbb{C}$ is complex differentiable. Then

$$\int_\gamma f(z)\,dz = 0$$

for every closed curve γ in U.

(b) State and prove Goursat's Theorem in detail.

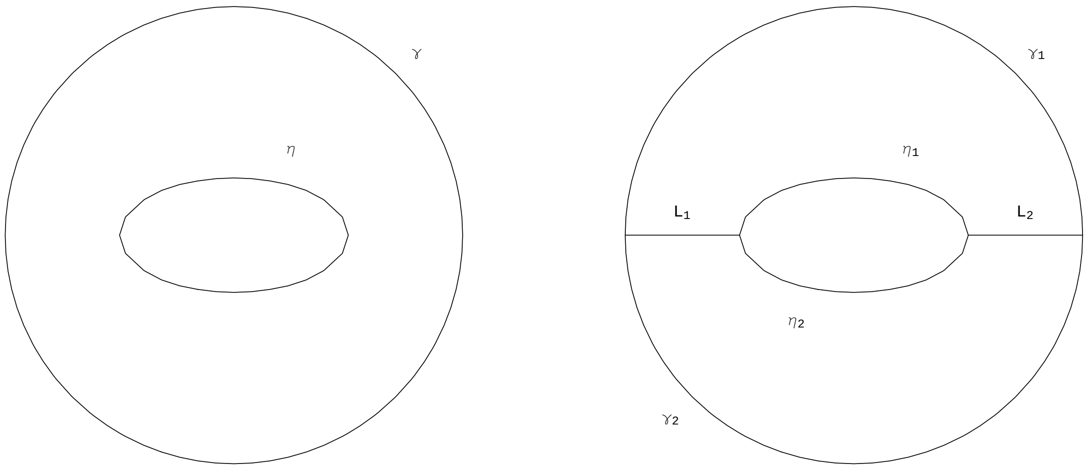
See the webpage's **Downloads** link for the complete proof.

(c) State and sketch the proof of the homotopy version of Cauchy's Integral Theorem.

The homotopy version states the following: Suppose γ and η are two positively oriented Jordan curves, with $\eta \subset \text{inside}(\gamma)$. If f is $\mathbb{C}$-differentiable on the region between the curves, i.e. f is $\mathbb{C}$-differentiable at every point on γ, η, and the set $\text{inside}(\gamma) \cap \text{outside}(\eta)$, then

$$\int_\gamma f(z)\,dz = \int_\eta f(z)\,dz.$$

As proof, suppose we have the setup shown below, with γ and η oriented positively (counterclockwise). Connect γ and η by two different lines, say L_1 and L_2. Suppose both are oriented from γ to η (outside to inside), and that we subdivide γ into γ_1, γ_2 and η into η_1, η_2 as indicated.



Then both $\sigma_1 := \gamma_1 + L_1 - \eta_1 - L_2$ and $\sigma_2 := \gamma_2 + L_2 - \eta_2 - L_1$ are Jordan curves, and f is holomorphic on a (simply connected) neighborhood of them. Hence, by the Cauchy Integral Formula, $\int_{\sigma_j} f\,dz = 0$ for $j = 1, 2$. But then

$$0 = \int_{\sigma_1} f(z)\,dz + \int_{\sigma_2} f(z)\,dz = \int_\gamma f(z)\,dz - \int_\eta f(z)\,dz,$$

as desired.

Problem 3. *Calculate the three line integrals below. You may use any of the Integral Theorems from class, but be sure to state which you are using and why it is justified!*

(a) $\int_L z\, e^{z^2}\, dz$, *where* L *is straight line from* 0 *to* $1+2i$.

The direct method is to parametrize L by $z(t) = t(1+2i)$ for $0 \le t \le 1$. Then

$$\int_L z\, e^{z^2}\, dz = \int_0^1 z(t)\, e^{z(t)^2}\, z'(t)\, dt = \int_0^1 (1+2i)^2 t\, e^{t^2(1+2i)^2}\, dt.$$

Under the substitution $u = t^2(1+2i)^2$, this becomes

$$\int_L z\, e^{z^2}\, dz = \int_0^{(1+2i)^2} \frac{1}{2} e^u\, du = \frac{1}{2}\left(e^{(1+2i)^2} - 1\right) = \frac{e^{-3+4i} - 1}{2}.$$

A faster method is to recognize that that $\frac{1}{2}e^{z^2}$ is a complex antiderivative of $z\, e^{z^2}$, whence the Fundamental Theorem implies

$$\int_L z\, e^{z^2}\, dz = \frac{1}{2}e^{z^2}\Big|_0^{1+2i} = \frac{1}{2}\left(e^{(1+2i)^2} - 1\right) = \frac{e^{-3+4i} - 1}{2}.$$

(b) $\int_C \frac{\sin(z^3)}{z-i}\, dz$, *where* C *is the positively-oriented circle* $\{|z| = 4\}$.

This is a straightforward application of Cauchy's Integral Formula. Since $\sin(z^3)$ is everywhere-holomorphic, the CIT implies

$$\sin(i^3) = \frac{1}{2\pi i}\int_C \frac{\sin(z^3)}{z-i}\, dz,$$

so

$$\int_C \frac{\sin(z^3)}{z-i}\, dz = 2\pi i\, \sin(-i) = 2\pi i\, \frac{e^{i(-i)} - e^{-i(-i)}}{2i} = \pi(e - e^{-1}).$$

(c) $\int_S \frac{2z+1}{z(z-1)^3}\, dz$, *where* S *is the positively-oriented square with sides along the lines* $x = \pm 2$ *and* $y = \pm 2$.

Observe that the integrand fails to be differentiable at $z = 0$ and $z = 1$. By the homotopy version of the CIT, the integral over S is equivalent to the integral over two circles, centered at 0 and 1, each of radius $1/2$. Now, using the CIF again, we have

$$\int\limits_{|z|=1/2} \frac{2z+1}{z(z-1)^3}\, dz = \int\limits_{|z|=1/2} \frac{\frac{2z+1}{(z-1)^3}}{z}\, dz = 2\pi i\, \frac{2z+1}{(z-1)^3}\Big|_{z=0} = -2\pi i,$$

while

$$\int\limits_{|z-1|=1/2} \frac{2z+1}{z(z-1)^3}\, dz = \int\limits_{|z-1|=1/2} \frac{\frac{2z+1}{z}}{(z-1)^3}\, dz = \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left\{\frac{2z+1}{z}\right\}\Big|_{z=1} = \pi i\, \frac{2}{z^3}\Big|_{z=1} = 2\pi i.$$

Thus

$$\int_S \frac{2z+1}{z(z-1)^3}\, dz = \int\limits_{|z|=1/2} \frac{2z+1}{z(z-1)^3}\, dz + \int\limits_{|z-1|=1/2} \frac{2z+1}{z(z-1)^3}\, dz = -2\pi\, i + 2\pi\, i = 0.$$

Problem 4. *Consider the real-valued function* $u(x, y) := x^2 - y^2 + x + 1$.

(a) Give the definition of a harmonic function. How does this condition relate to complex differentiability?

A real-valued function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if it is of class C^2 and satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$$

on U.

Loosely speaking, harmonic functions are the real and/or imaginary parts of a holomorphic function. More precisely, if $f = u + i\,v$ is holomorphic on U, then both u and v are harmonic on U. Conversely, if U is simply connected and u is harmonic on U, then there exists a holomorphic function f on U such that $u = \operatorname{Re} f$.

(b) Show that u *is harmonic on* $\mathbb{C}$.

Note that u is a polynomial, so u is certainly C^2. (In fact, it is C^∞). As for Laplace's equation, note that

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2,$$

so $u_{xx} + u_{yy} = 2 - 2 = 0$, and u is harmonic.

(c) Find the harmonic conjugate v *of* u.

We seek a harmonic function v that satisfies the CR equations with u. Suppose $v(x, y)$ is such a function. Then

$$v(x, y) = \int \frac{\partial v}{\partial x}\, dx = \int -\frac{\partial u}{\partial y}\, dx = \int 2y\, dx = 2xy + C(y).$$

But then

$$2x + C'(y) = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + 1,$$

whence $C'(y) = 1$, so (up to a constant) $C(y) = y$. Thus, we must have

$$v(x, y) = 2xy + y.$$

Since this is easily seen to be harmonic (indeed, $v_{xx} = v_{yy} = 0$), this is a harmonic conjugate to u.

(d) Explain why the function $h(x, y) = \frac{1}{2}\ln(x^2 + y^2)$ *has no harmonic conjugate on* $\mathbb{C} \setminus \{0\}$.

To show this is impossible, assume to the contrary a conjugate $v(z)$ does exists on $\mathbb{C}\setminus\{0\}$. For convenience, let use denote by $\mathbb{C}^*$ the punctured plane $\mathbb{C} \setminus \{0\}$ and by $\mathbb{C}_C$ denote the cut-domain $\mathbb{C} \setminus \{\text{negative axis}\}$.

On one hand, since v is a harmonic conjugate, the function $f(z) := u(z) + i\,v(z)$ is defined and holomorphic on $\mathbb{C}^*$. On the other hand, the principal branch $\operatorname{Log}(z) = \ln(|z|) + i\, \arg_p(z)$ is defined and holomorphic on $\mathbb{C}_C$, and its real part agrees with u. Thus, the function

$$z \mapsto f(z) - \operatorname{Log}(z)$$

is holomorphic on $\mathbb{C}_C$ and takes on purely imaginary values. This implies it is a constant, whence

$$\operatorname{Log}(z) \equiv f(z) - iK, \qquad \forall\, z \in \mathbb{C}_C$$

for some constant K. But since f is holomorphic on $\mathbb{C}^*$, this implies we can define the principal branch of the logarithm to be holomorphic on $\mathbb{C}^*$ as well; however, we know that the principal branch of the logarithm is necessarily discontinuous at any negative real number. This contradiction implies no harmonic conjugate v can exist on $\mathbb{C}^*$.

Problem 5. *Choose one of the following theorems below and (a) state the result and give a detailed, complete proof, (b) discuss the relevance of the result in complex analysis, and (c) give at least one useful application of the result.*

- *The equivalence of $\mathbb{C}$-diff and the Cauchy-Riemann equations*
- *The existence of complex antiderivatives (the FTC part II)*
- *The Cauchy Estimates & Liouville's Theorem*

See the webpage's **Downloads** link for complete proofs of the various theorems. As for relevance and application:

- *Equivalence.* The revelance is in relating a purely complex condition to two purely real conditions. This allows us to determine differentiability not by going to the complex limit (hard), but by instead calculating partial derivatives (easy) and verifying the CR equations (easy).

- *FTC Part II.* The relevance is two-fold. First, it extends the real FTC to the complex realm, giving a means to calculate antiderivatives. Second, it illustrates an important difference between the real and complex cases: in the real world, continuity is sufficient to guarantee an antiderivative, but not in the complex world. As applications, it allows for the quick evaluation of line integrals if a suitable antiderivative can be easily found.

- *Cauchy estimates & Liouville's Theorem.* The revelance of the estimates and Liouville's Theorem is that they are uniqueness results: they help give conditions useful in restricting the space of holomorphic functions. An application of the pair is the Fundamental Theorem of Algebra, which is an almost immediate consequence of Liouville's Theorem, which is itself an immediate consequence of the Estimates.