Midterm Solutions, Part 2

Problem 1. *In this problem we shall prove the following: Suppose U is a simply connected domain and $f : U \to \mathbb{C}$ is continuous and holomorphic at every point in U except possibly at ζ_0. Then f is holomorphic at ζ_0.*

(a) Step 1. Carefully prove using ϵ's and δ's that

$$\lim_{R\to 0} \int_{|z-\zeta_0|=R} f(z)\,dz = 0.$$

Let $\epsilon > 0$. We must find a $\delta > 0$ such that

$$0 < R < \delta \quad \Longrightarrow \quad \left| \int_{|z-\zeta_0|=R} f(z)\,dz \right| < \epsilon.$$

Since U is open, there exists an $R_{\max}$ such that

$$B(\zeta_0, R_{\max}) \subset U.$$

Moreover, since f is continuous at ζ_0, there exists $\eta > 0$ such that

$$|z-\zeta_0| < \eta \quad \Longrightarrow \quad \big|f(z) - f(\zeta_0)\big| < \frac{\epsilon}{2\pi}.$$

Let C_R denote the circle $|z-\zeta_0| = R$ in the positive direction. Observe that since the constant function 1 is everywhere holomorphic, Cauchy's Integral Formula implies $\int_\gamma 1\,dz = 0$ for any closed curve γ. Now, if $0 < r < R_{\max}$, we have

$$\int_{C_R} f(z) - f(\zeta_0)\,dz = \int_{C_R} f(z)\,dz - f(\zeta_0)\int 1\,dz = \int_{C_R} f(z)\,dz - 0 = \int_{C_R} f(z)\,dz,$$

Let us set $\delta := \min\{\eta, R_{\max}, 1\}$. Then if $0 < R < \delta$, we have

$$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_{C_R} f(z) - f(\zeta_0)\,dz\right| \le \int_{C_R} \big|f(z) - f(\zeta_0)\big|\,|dz| < \int_{C_R} \frac{\epsilon}{2\pi}\,|dz| = \frac{\epsilon}{2\pi} 2\pi R < \epsilon,$$

which completes the proof.

(b) Step 2. Use this and Cauchy's Integral Theorem to show that f is holomorphic at ζ_0.

We give a sketch of the proof. From our equivalences in class, if we can show that f is conservative on U, then f is necessarily holomorphic on U and, in particular, at ζ_0. Thus, we shall show that

$$\int_\gamma f(z)\,dz = 0$$

for any Jordan curve γ in $\mathbb{C}$.

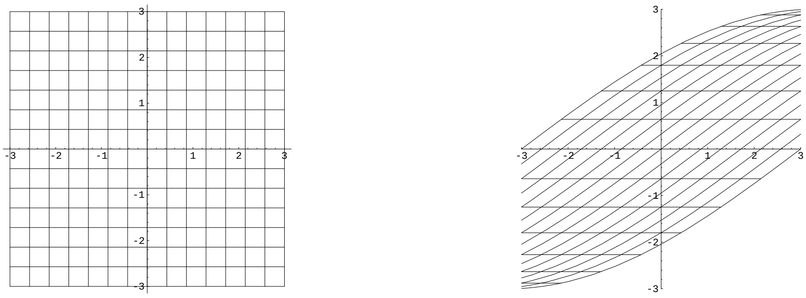
Consider such a curve γ. If ζ_0 is on the outside of γ, then because the domain containing γ is simply connected and f is holomorphic on it, we have $\int_\gamma f\,dz = 0$ by Cauchy's Theorem. On the other hand, if ζ_0 lies inside the triangle, then $\int_\gamma f\,dz = 0$ from part *(a)* and the homotopy version of CIT.

Lastly, if ζ_0 lies on the curve, then we can approximate the integral $\int_\gamma f\,dz$ by a polygonal path P which doesn't include ζ. But again, either ζ_0 lies inside the polygon P or outside, and in either case we've already shown $\int_P f\,dz = 0$, so we must conclude $\int_\gamma f\,dz = 0$ as well.

(c) Does this theorem hold for real-valued functions $f : (a,b) \to \mathbb{R}$?

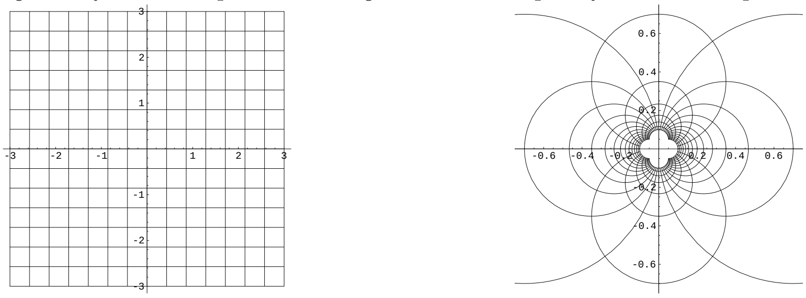
No. Consider $f(x) = |x|$. This is continuous on $\mathbb{R}$ and differentiable for every $x \neq 0$, but it fails to be differentiable at $x = 0$.

Problem 2. *(a) Can the following transformation picture belong to a holomorphic function? Explain why or why not.*



No, it cannot. Holomorphic functions are conformal, i.e. angle preserving. This mapping takes right angles and pinches them into acute angles, and so it cannot be holormorphic.

(b) Can the following transformation picture belong to a holomorphic function? Explain why or why not.



Yes, this can be a holomorphic mapping. Not only does it appear to preserve right angles, it also appears to send the straight lines on the left to circles on the right, which suggests that it might actually be a Möbius transformation. (In fact, this is the function $1/z$, which is holomorphic away from $z = 0$.

(c) Suppose that $f : U \to \mathbb{C}$ is holomorphic with $f(2-i) = 3$ and $f'(2-i) = -2i$. Make an educated guess — i.e. approximate — $f(\frac{3}{2} - i)$ and $f(2 - \frac{i}{2})$.

Near $f(2-i) = 3$, f acts as an amplitwist by $f'(2-i) = -2i$. That is, tangent directions at $2-i$ are sent to tangent directions at 3 as if by an amplitwist of $-2i$, i.e. dilation by 2 and rotation by 90° clockwise.

Now, the point $\frac{3}{2} - i$ is found by moving from $2-i$ in the tangent direction of $-\frac{1}{2}$; hence, the point $f(\frac{3}{2} - i)$ should be found by moving from $f(2-i)$ by the appropriate amplitwist $(-2i) \times (-\frac{1}{2}) = i$. Thus,

$$f\left(\frac{3}{2} - i\right) \approx f(2-i) + i = 3 + i.$$

Similarly, the point $2 - \frac{i}{2}$ is found by moving from $2-i$ in the tangent direction of $\frac{i}{2}$; hence, the point $f(2 - \frac{i}{2})$ should be found by moving from $f(2-i)$ by the appropriate amplitwist $(-2i) \times (\frac{i}{2}) = 1$. Thus,

$$f\left(2 - \frac{i}{2}\right) \approx f(2-i) + 1 = 4.$$

Problem 3. *Suppose that U is an open domain (not necessarily simply-connected), and $f : U \to \mathbb{C}$ is holomorphic. If $\lvert f(z) - 1\rvert < 1$ for all $z \in U$, then prove that*

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for any closed curve γ in U.

It suffices to show that the function $f'(z)/f(z)$ has a complex antiderivative $F(z)$ defined on the set U, for then given any closed curve, the Fundamental Theorem implies

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = F\big(\gamma_{\text{end}}\big) - F\big(\gamma_{\text{start}}\big) = 0$$

since γ is closed.

To define an antiderivative, let us denote by $\mathbb{C}_C$ the cut-domain $\mathbb{C} \setminus \{\text{negative axis}\}$, and consider the principal branch of the logarithm

$$\operatorname{Log}(z) := \ln(|z|) + i\ \arg_p(z), \qquad -\pi < \arg_p(z) < \pi,$$

which is holomorphic on $\mathbb{C}_C$.

Since $|f(z) - 1| < 1$ for every $z \in U$, it follows that $f(z) \neq 0$, so the function $f'(z)/f(z)$ is well defined and holomorphic in U. In fact, we actually have that

$$z \in U \quad \Longrightarrow \quad f(z) \in B(1,1) \subset \mathbb{C}_C,$$

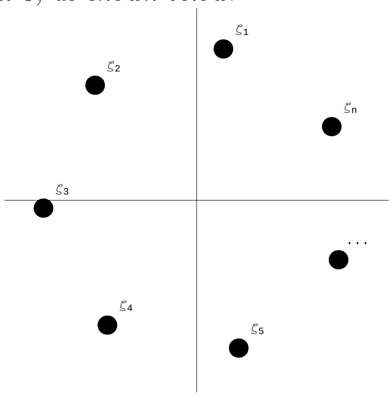
which means that the composition function $\operatorname{Log} f(z)$ is well-defined and holomorphic.

We claim that $F(z) := \operatorname{Log} f(z)$ is the desired antiderivative. Indeed, by the Chain Rule we have

$$F'(z) = \frac{d}{dz}\big\{\operatorname{Log} f(z)\big\} = \frac{1}{f(z)}\, f'(z) = \frac{f'(z)}{f(z)},$$

which completes the proof.

Problem 4. *Suppose that $\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_n$ are n complex numbers situated in such a way that they form the vertices of a regular n-gon centered at 0, as shown below:*



(a) Prove that $(\zeta_1)^n = (\zeta_2)^n = (\zeta_3)^n = \cdots = (\zeta_n)^n$.

Since the points ζ_j form a regular n-gon, it follows that each is the same fixed distance r_0 from the origin. Moreover, the angles between consecutive points must be the same, and in fact, must be $2\pi/n$.

Now, write ζ_1 in polar coordinates as $r_0\, e^{i\theta_0}$. Since ζ_2 has the same modulus and is found by rotating through an angle of $2\pi/n$ counterclockwise, we have

$$\zeta_2 = r_0\, e^{i(\theta_0 + 2\pi/n} = \big(r_0\, e^{i\theta_0}\big)e^{2\pi\, i/n} = \zeta_1\, e^{2\pi\, i/n}.$$

Repeating by induction, we see that

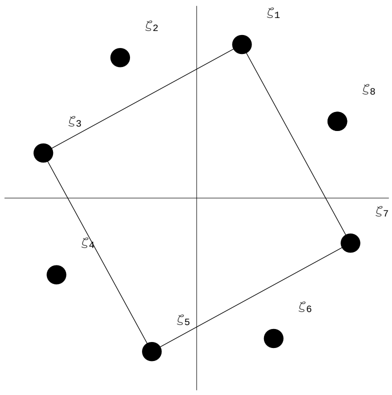
$$\zeta_k = \zeta_{k-1}\, e^{2\pi\, i/n} = \zeta_1 \big(e^{2\pi\, i/n}\big)^{k-1}, \qquad 1 \le k \le n.$$

Hence,

$$(\zeta_k)^n = \Big(\zeta_1 \big(e^{2\pi\, i/n}\big)^{k-1}\Big)^n = (\zeta_1)^n \Big(\big(e^{2\pi\, i/n}\big)^{k-1}\Big)^n = (\zeta_1)^n \big(e^{2\pi i}\big)^{k-1} = (\zeta_1)^n \cdot 1 = (\zeta_1)^n.$$

(b) Let η be the closed polygonal path joining $\zeta_1 \to \zeta_3 \to \zeta_5 \to \cdots \to \zeta_1$. Calculate $\int_\eta z^{-1}\, dz$.

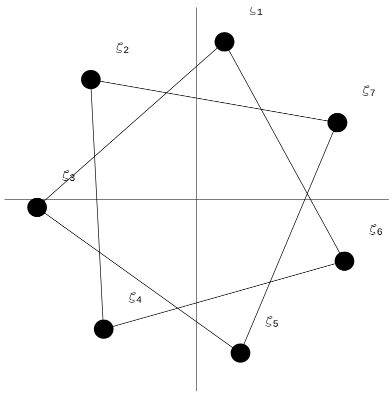
Suppose n is even and $n > 2$ (lest the path pass through the origin). Observe that if there are an even number of points, then η connects exactly half of them, entirely missing the even-subscripted terms, as illustrated below.



As a result, η forms a complete clockwise loop around the origin. In particular, η is a Jordan curve, so by the Cauchy Integral Formula,

$$\int_\eta \frac{1}{z}\, dz = 2\pi i \{1\}\Big|_{z=0} = 2\pi i.$$

Suppose instead that n is odd and $n > 3$ (explanation in a moment). Observe that if there are an odd number of points, then η connects all of the points while wrapping around the curve twice. Hence, the curve resembles a star, while all orientations directed in the clockwise orientation.



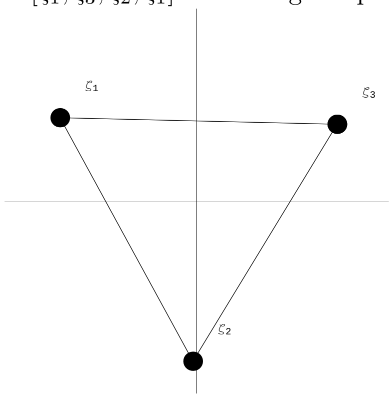
In this case, η is *not* a Jordan curve, since it intersects itself n times, so we cannot immediately apply the Cauchy Integral Formula. However, we can decompose η into *two* Jordan curves — the small n-gon on the inside and the n-pointed star on the outside — both oriented positively. Applying the Cauchy Integral Formula to either polygon yields

$$\int_{\text{polygon}} \frac{1}{z}\,dz = 2\pi i\{1\}\Big|_{z=0} = 2\pi i,$$

whence

$$\int_\eta \frac{1}{z}\,dz = 4\pi i.$$

Lastly, if $n = 3$, then note that $\eta = [\zeta_1, \zeta_3, \zeta_2, \zeta_1]$ is a triangular path oriented clockwise, i.e. negatively.



But then $-\eta$ is positively-oriented, whence

$$\int_{-\eta} \frac{1}{z}\,dz = 2\pi i \quad \Longrightarrow \quad \int_\eta \frac{1}{z}\,dz = -2\pi i.$$

Problem 5. *Let* $f(z) = \frac{z-i}{z^2-1}$.

(a) Calculate $\int_S f(z)\,dz$*, where* S *is the left-half of the semicircle* $\{|z+1| = 1\}$ *from* $-1-i$ *to* $-1+i$.

By partial fractions, we have

$$\frac{z-i}{z^2-1} = \frac{1-i}{2}\,\frac{1}{z-1} + \frac{1+i}{2}\,\frac{1}{z+1},$$

whence

$$\int_S f(z)\,dz = \frac{1-i}{2}\int_S \frac{dz}{z-1} + \frac{1+i}{2}\int_S \frac{dz}{z+1}.$$

For the first integral, consider the function $F(z) = \log(z-1)$, where log is defined for arguments $0 < \theta < 2\pi$. Since $z-1$ stays in the left-half plane for $z \in S$, it follows that $F(z)$ is an antiderivative of $(z-1)^{-1}$, whence the Fundamental Theorem implies

$$\begin{aligned}\frac{1-i}{2}\int_S \frac{dz}{z-1} &= \frac{1-i}{2}\,\log(z-1)\Big|_{-1-i}^{-1+i} = \frac{1-i}{2}\big(\log(-2+i) - \log(-2-i)\big)\\ &= \frac{1-i}{2}\left[\ln\sqrt{5} + i\left(\pi - \tan^{-1}\frac{1}{2}\right) - \ln\sqrt{5} - i\left(\pi + \tan^{-1}\frac{1}{2}\right)\right]\\ &= \frac{1-i}{2}\left(-2i\,\tan^{-1}\frac{1}{2}\right) = -(1+i)\tan^{-1}\frac{1}{2}.\end{aligned}$$

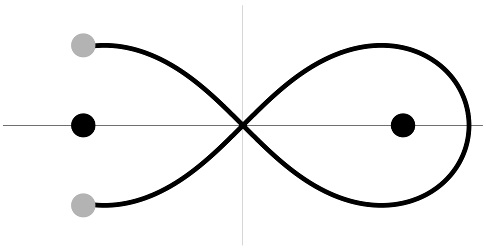
For the second integral, consider the function $G(z) = \log(z+1)$, where log is again defined for arguments $0 < \theta < 2\pi$. Since $z+1$ stays in the left-half plane for $z \in S$, it follows that $G(z)$ is an antiderivative of $(z+1)^{-1}$, whence the Fundamental Theorem implies

$$\begin{aligned}\frac{1+i}{2}\int_S \frac{dz}{z+1} &= \frac{1+i}{2}\,\log(z+1)\Big|_{-1-i}^{-1+i} = \frac{1+i}{2}\big(\log(i) - \log(-i)\big)\\ &= \frac{1+i}{2}\left(\frac{\pi i}{2} - \frac{3\pi i}{2}\right) = \frac{1+i}{2}\left(-\pi i\right) = \frac{\pi}{2}(1-i).\end{aligned}$$

Hence, the answer is

$$\int_S f(z)\,dz = \frac{\pi}{2}(1-i) - (1+i)\tan^{-1}\frac{1}{2}.$$

(b) Calculate $\int_\gamma f(z)\,dz$*, where* γ *be the curve below, which begins at* $-1-i$ *and ends at* $-1+i$.



Observe that $\gamma - S$ is a closed curve, where S is from part *(a)*. Consequently, we can break it up into two Jordan curves: σ_1, the positively-oriented teardrop on the left, and σ_2, the negatively-oriented teardrop on the right.

Using the Cauchy Integral formula,

$$\int_{\sigma_1} f(z)\,dz = \frac{1-i}{2}\int_{\sigma_1}\frac{dz}{z-1} + \frac{1+i}{2}\int_{\sigma_1}\frac{dz}{1+z} = \frac{1-i}{2}\cdot 0 + \frac{1+i}{2}\cdot 2\pi i = \pi(-1+i),$$

while

$$\int_{-\sigma_2} f(z)\,dz = \frac{1-i}{2}\int_{-\sigma_2}\frac{dz}{z-1} + \frac{1+i}{2}\int_{-\sigma_2}\frac{dz}{1+z} = \frac{1-i}{2}\cdot 2\pi i + \frac{1+i}{2}\cdot 0 = \pi(1+i),$$

whence

$$\int_{\sigma_2} f(z)\,dz = \pi(-1-i).$$

Thus, it follows that the integral over the closed curve is

$$\pi(-1+i) + \pi(-1-i) = -2\pi,$$

whence

$$\int_\gamma f(z)\,dz = -2\pi - \int_S f(z)\,dz = -2\pi - \frac{\pi}{2}(1-i) + (1+i)\tan^{-1}\frac{1}{2} = \frac{\pi(-5+i)}{2} + (1+i)\tan^{-1}\frac{1}{2}.$$

Problem 6. *(a) Find the Möbius transformation that sends*

$$2 \mapsto 1, \qquad 4i \mapsto \infty, \qquad -4i \mapsto 0.$$

Let us write

$$m(z) = \frac{az+b}{cz+d}.$$

The condition $m(-4i) = 0$ implies $-4ia + b = 0$, whence $b = 4i$. Similarly, the condition $m(4i) = 0$ implies $4ic + d = 0$, whence $d = -4i$. Hence,

$$m(z) = \frac{a(z+4i)}{c(z-4i)} = k\frac{z+4i}{z-4i}.$$

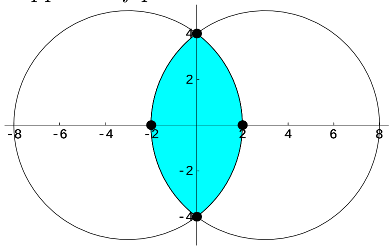
Finally, then condition $m(2) = 1$ implies

$$1 = m(2) = k\frac{2+4i}{2-4i} = k\left(\frac{-3+4i}{5}\right) \quad \Longrightarrow \quad k = \frac{5}{-3+4i} = -\frac{3+4i}{5},$$

whence

$$m(z) = \left(-\frac{3+4i}{5}\right)\left(\frac{z+4i}{z-4i}\right) = \frac{3+4i}{5}\,\frac{4i+z}{4i-z}.$$

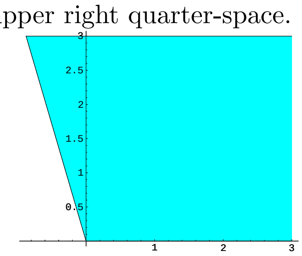
(b) Consider the open domain $D = B(-3,5) \cap B(3,5)$, shown below. Find a holomorphic function that carries this region bijectively onto the upper half-plane.



Since m sends the points $(-4i, 2, 4i)$ to $(0, 1, \infty)$ in that order, m sends the right circular arc of D onto the positive real axis. On the other hand, since

$$m(-2) = \frac{-7+24i}{25}, \qquad m(0) = \frac{3+4i}{5},$$

we conclude that m sends the left circular arc to the line through 0 and $\frac{-7+24i}{25}$, i.e. the line $y = -\frac{24}{7}x$, and the region between the circles into the upper right quarter-space.



Observe that these two lines meet at an angle of $\pi - \tan^{-1}\frac{24}{7}$. Hence, if we compose this mapping with the (principal branch of the) power function

$$z \mapsto z^{i\pi/(\pi - \tan^{-1}(24/7))},$$

then the skewed line (in the direction of $e^{i(\pi - \tan^{-1}(24/7))}$) will be sent to the negative real axis (in the direction $e^{i\pi}$), and the region to the upper half plane. Hence, we can take as our function

$$H(z) = \left[\frac{3+4i}{5}\,\frac{4i+z}{4i-z}\right]^{i\pi/(\pi - \tan^{-1}(24/7))}.$$