Midterm Part 2, Problem 5

Here is a second solution for Problem 5, Part *(a)*, which was suggested by an idea from RV.

Problem 5, part (a)
Calculate $\int_S \frac{z-i}{z^2-1}\,dz$*, where* S *is the left half of the semicircle* $\{|z+1| = 1\}$ *from* $-1 = i$ *to* $1+i$*.*

Antiderivative solution. This is essentially the version that appears in the online solutions, slightly rewritten.

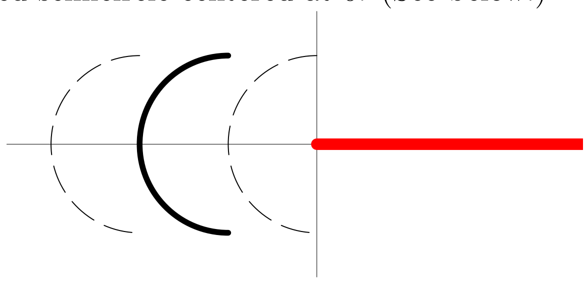
Observe that using partial fractions, we can rewrite the integrand as

$$\frac{z-i}{z^2-1} = \left(\frac{1+i}{2}\right)\frac{1}{z+1} + \left(\frac{1-i}{2}\right)\frac{1}{z-1}.$$

Consider the branch of the logarithm defined by cutting out the positive real axis, i.e.

$$\log(z) := \ln|z| + i\ \arg(z), \qquad 0 < \arg(z) < 2\pi.$$

Observe that if $z \in S$, then $z-1$ lies on a left-handed semicircle centered at -2, whereas $z+1$ lies on a left-handed semicircle centered at 0. (See below.)



In either case, $z \pm 1$ does not cross the branch cut of the positive axis, so these values lie in the domain of our logarithm.

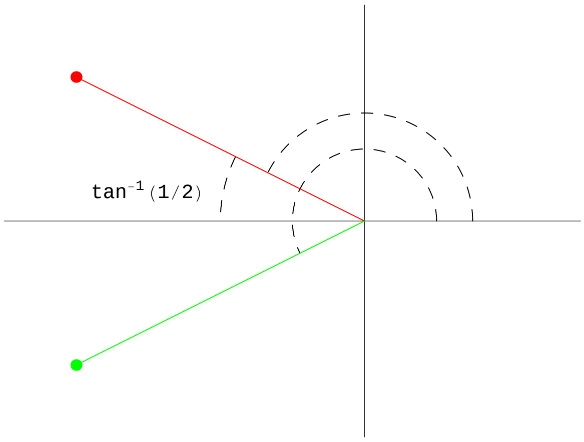
Now, observe that $\log(z+1)$ is therefore a well-defined antiderivative of $(z+1)^{-1}$ on a neighborhood of S, whence the Fundamental Theorem implies

$$\int_S \frac{dz}{z+1} = \log(z+1)\Big|_{-1-i}^{-1+i} = \log(i) - \log(-i) = \frac{\pi\,i}{2} - \frac{3\pi\,i}{2} = -\pi\,i.$$

Similarly, $\log(z-1)$ is therefore a well-defined antiderivative of $(z-1)^{-1}$ on a neighborhood of S, whence the Fundamental Theorem implies

$$\begin{aligned}\int_S \frac{dz}{z-1} &= \log(z-1)\Big|_{-1-i}^{-1+i} = \log(-2+i) - \log(-2-i)\\ &= \left[\ln\sqrt{5} + i\left(\pi - \tan^{-1}\frac{1}{2}\right)\right] - \left[\ln\sqrt{5} + i\left(\pi + \tan^{-1}\frac{1}{2}\right)\right] = 2i\ \tan^{-1}\frac{1}{2}.\end{aligned}$$

(See the figure below for the geometry of these numbers.)



Hence,

$$\int_S \frac{z-i}{z^2-1}\,dz = \left(\frac{1+i}{2}\right)\big(-\pi\, i\big) + \left(\frac{1-i}{2}\right)\left(-2i\,\tan^{-1}\frac{1}{2}\right) = \frac{\pi}{2}(1-i) - (1+i)\tan^{-1}\frac{1}{2}.$$

Parametrization solution. Again we write

$$\frac{z-i}{z^2-1} = \left(\frac{1+i}{2}\right)\frac{1}{z+1} + \left(\frac{1-i}{2}\right)\frac{1}{z-1}.$$

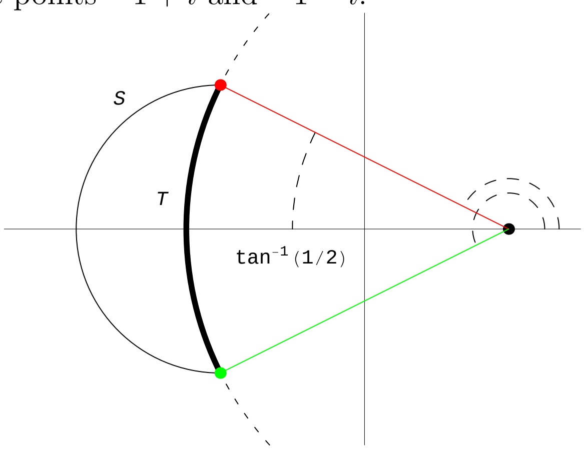
For the first integral, let us parametrize S by the path

$$z(t) := -1 + e^{it}, \qquad \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$$

Observe, however, that this parametrizes the curve in the wrong direction. Thus,

$$\int_S \frac{1}{z+1}\,dz = -\int_{\pi/2}^{3\pi/2} \frac{1}{e^{it}} i\, e^{it}\,dt = -\int_{\pi/2}^{3\pi/2} i\,dt = -\pi\, i.$$

For the second integral, consider *not* the semicircle S but the arc T of the circle centered at 1 passing through the points $-1+i$ and $-1-i$.



Since $(z-1)^{-1}$ is holomorphic on the region between S and T, the Cauchy Integral Theorem implies that the integrals over S and T are the same. Now, we can parametrize T by

$$z(t) = 1 + \sqrt{5}\, e^{it}, \qquad \pi - \tan^{-1}\frac{1}{2} \le t \le \pi + \tan^{-1}\frac{1}{2}.$$

Again, this parametrizes T in the wrong direction, whence

$$\int_T \frac{1}{z+1}\, dz = -\int_{\pi-\tan^{-1}(1/2)}^{\pi+\tan^{-1}(1/2)} \frac{1}{\sqrt{5}\, e^{it}} i\sqrt{5}\, e^{it}\, dt = -\int_{\pi-\tan^{-1}(1/2)}^{\pi+\tan^{-1}(1/2)} i\, dt = -2i\, \tan^{-1}\frac{1}{2}.$$

Thus, as before,

$$\int_S \frac{z-i}{z^2-1}\, dz = \left(\frac{1+i}{2}\right)\big(-\pi i\big) + \left(\frac{1-i}{2}\right)\left(-2i\, \tan^{-1}\frac{1}{2}\right) = \frac{\pi}{2}(1-i) - (1+i)\tan^{-1}\frac{1}{2}.$$