Convergence in $\mathbb{C}$

- **Complex norm.** The modulus $|z|$ is precisely the Euclidean norm in $\mathbb{R}^2$ and satisfies two important properties:

$$|zw| = |z||w|, \qquad \big||z| - |w|\big| \le |z + w| \le |z| + |w|.$$

- **Complex distance.** The *distance* between complex numbers z and w is defined by

$$\operatorname{dist}(z, w) := |z - w|,$$

which is precisely their Euclidean distance in $\mathbb{R}^2$. For a point $z \in \mathbb{C}$, the *neighborhood* of z of radius $\delta > 0$ is

$$B(z, \delta) := \{w \in \mathbb{C} : |w - z| < \delta\}.$$

- **Convergence.** A sequence of complex numbers (z_n) *converges* to the number $\zeta \in \mathbb{C}$, denoted

$$z_n \to \zeta \quad \text{as} \quad n \to \infty \qquad \text{or} \qquad \lim_{n\to\infty} z_n = \zeta,$$

if the sequence is eventually contained completely in any neighborhood of ζ, i.e.

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n \ge N \Longrightarrow |z_n - \zeta| < \epsilon.$$

The number ζ is called *the limit* of the sequence. The usual rules apply to sums, differences, products, and quotients. Moreover, it is useful to note that $z_n \to \zeta$ iff $\operatorname{Re} z_n \to \operatorname{Re} \zeta$ and $\operatorname{Im} z_n \to \operatorname{Im} \zeta$.

- **Cauchy sequences.** A sequence $(z_n) \subset \mathbb{C}$ is called *Cauchy* if its terms eventually get arbitrarily close together, i.e.

$$\forall \epsilon > 0\, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n, m \ge N \Longrightarrow |z_n - z_m| < \epsilon.$$

Since $\mathbb{C}$ is a complete metric space, every Cauchy sequence converges to some complex number.

- **Limit points.** A point $\lambda \in \mathbb{C}$ is *a limit point* of the sequence (z_n) if the sequence is frequently in any neighborhood of λ, i.e.

$$\forall \epsilon > 0\, \exists \text{ infinitely many } n \quad \text{s.t.} \quad |z_n - \lambda| < \epsilon.$$

If λ is a limit point of (z_n), then some subsequence (z_{n_j}) converges to λ. Moreover, the Bolzano Theorem implies any bounded sequence must have a limit point. Note that a sequence can have several *limit points*, but only one *limit*.