## The topology of $\mathbb{C}$

Given a set $E \subset \mathbb{C}$ and a point $z \in \mathbb{C}$:

- $z$ is in the *interior* of $E$, denoted $z \in E^\circ$, if $z$ has a neighborhood contained in $E$.
- $z$ is in the *exterior* of $E$ if $z$ has a neighborhood disjoint from $E$.
- $z$ is on the *boundary* of $E$, denoted $z \in \partial E$, if every neighborhood of $z$ intersects $E$ and its complement.
- $E$ is *open* if every point of $E$ is an interior point, i.e. $E = E^\circ$.
- $E$ is *closed* if $E$ contains its boundary, i.e. $\partial E \subset E$.
- The *closure* of $E$ is the closed set $\overline{E} := E \cup \partial E$.
- $E$ is *connected* if there does not exist disjoint open sets $U, V$ such that the following holds: $E \subset U \cup V$, $E \cap U \neq \emptyset$, and $E \cap V \neq \emptyset$.
- $E$ is *compact* if every covering of $E$ by neighborhoods admits a finite subcovering. Equivalently, by the Heine-Borel Theorem, $E$ is closed and bounded.
- $E$ is a *domain* if $E$ is connected and open. Any set $R$ such that $E \subset R \subset \overline{E}$ is called a *region.*
- A *path* in $E$ is a continuous vector-valued function $\sigma : [a, b] \to E \subset \mathbb{R}^2$. The image of the path is called an *arc* (or *curve*) in $E$. If in addition $\sigma(a) = \sigma(b)$, the curve is called *closed.*
- $E$ is *arcwise connected* if for any points $z, w \in E$ there exists an arc with endpoints at $z$ and $w$. Note that every arcwise connected set is connected; the converse holds if the set is also open.
- A closed curve $\sigma : [a, b] \to \mathbb{C}$ that satisfies the additional condition that $\sigma(t_1) = \sigma(t_2)$ iff $t_1 = a$ and $t_2 = b$ is called a *Jordan curve.* The Jordan Curve Theorem states that the complement of such a curve consists of two domains, one bounded (called the *inside* of the curve), and one unbounded (called the *outside*).
- A connected set $E$ is called *simply connected* if the inside of every Jordan curve in $E$ is contained in $E$. Intuitively, a set is simply connected if it contains no holes or punctured points.

It is worthwhile to note that the complex topology of $\mathbb{C}$ is *precisely* the Euclidean topology of $\mathbb{R}^2$. Hence, any topological theorem about $\mathbb{R}^2$ is also a topological theorem (under a change of notation) for $\mathbb{C}$.

## Complex functions $\mathbb{C}$

- **Complex functions.** A complex function $f$ on a set $E \subset \mathbb{C}$, denoted $f : E \to \mathbb{C}$, is a rule which assigns to each complex number $z \in E$ a unique complex number $f(z) \in \mathbb{C}$. The set $E$ is called the *domain of definition* of $f$.

- **Complex functions as mappings.** Notice that any complex function is the sum of two real-valued functions on $E$,

$$u(z) := \operatorname{Re} f(z), \qquad v(z) := \operatorname{Im} f(z).$$

Hence, any complex function $f = u + i\,v$ can be viewed as a vector field $f = (u, v)$ on $\mathbb{R}^2$, i.e. a vector-valued mapping defined on a subset of $\mathbb{R}^2$.

- **Visualizing complex functions.** There are four main ways to visualize a complex function $f = u + i\,v$:

  - *Transformation picture:* sketch two copies of the complex plane showing how a general domain is transformed under the function $f$.
  - *Vector field plot:* in the plane, draw at each point $z$ the vector $f(z)$. This is useful in physics applications.
  - *Modular plot:* in 3-space, graph the real-valued function $|f(z)|$. This is useful when checking continuity.
  - *Graph pairs:* in 3-space, plot separately the graphs of $u(z)$ and $v(z)$.

- **Limits.** We say that a complex function $f : E \to \mathbb{C}$ *approaches* $A$ at the point $\zeta \in \mathbb{C}$, denoted

$$f(z) \to A \quad \text{as} \quad z \to \zeta \qquad \text{or} \qquad \lim_{z \to \zeta} f(z) = A,$$

if the values of $f$ can be made arbitrarily close to $A$ provided $z$ is sufficiently close to $\zeta$, i.e.

$$\forall \epsilon > 0\, \exists \delta > 0 \quad \text{s.t.} \quad 0 < |z - \zeta| < \delta,\, z \in E \Longrightarrow \big|f(z) - A\big| < \epsilon.$$

This is equivalent to the following sequential formulation: the sequence $f(z_n)$ converges to $A$ for every sequence $(z_n) \subset E$ which converges to $\zeta$.

- **Continuity.** A function $f : E \to \mathbb{C}$ is *continuous* at the point $z_0 \in E$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

If $f$ is continuous at every point of $E$, then $f$ is *continuous in* $E$. Observe that the complex function $f$ is continuous at $z_0$ if and only both real functions $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at $z_0$.

## Complex infinity

- **The extended plane.** The *extended complex plane* is the set $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The set

$$B(\infty, \delta) := \left\{ z \in \mathbb{C} : |z| > \frac{1}{\delta} \right\} \cup \{\infty\}$$

is called the *neighborhood* of infinity of radius $\delta$. This set without the point $\infty$ – which is a subset of $\mathbb{C}$ – is called a *deleted neighborhood* of infinity, and is also denoted by $B(\infty, \delta)$. Observe that this implies that $\mathbb{C}_\infty$ is the *one-point compactification* of $\mathbb{C}$.

- **Sequences in the extended plane.** A sequence $(z_n)$ in $\mathbb{C}_\infty$ *converges* to the point $\zeta \in \mathbb{C}_\infty$ if

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \Longrightarrow z_n \in B(\zeta, \delta).$$

Observe that if $\zeta \in \mathbb{C}$, then this definition coincides with the standard definition. On the other hand, if $\zeta = \infty$, this means that the terms $|z_n|$ eventually grow arbitrarily large.

- **Limits in the extended plane.** We say that a function $f : E \to \mathbb{C}$ *approaches* $A \in \mathbb{C}_\infty$ at the point $\zeta \in \mathbb{C}_\infty$ if

$$\forall \epsilon > 0 \, \exists \delta > 0 \quad \text{s.t.} \quad z \in B(\zeta, \delta),\, z \in E \setminus \{\zeta\} \Longrightarrow f(z) \in B(A, \epsilon).$$

Observe that if $\zeta \in \mathbb{C}$, then this definition coincides with the standard definition.

However, if $\zeta = \infty$, this means that the values $f(z)$ are arbitrarily close to $A$ if $|z|$ is sufficiently large; in this case, we write $f(\infty) = A$. Similarly, if $A = \infty$, this means that $|f(z)|$ is arbitrarily large is $z$ is sufficiently close to $\zeta$; in this case, we write $f(\zeta) = \infty$.

- **The Riemann Sphere.** The set $\mathbb{C}_\infty$ is homeomorphic to the unit sphere $S^2$. An explicit homeomorphism is given by the stereographic projection of the sphere $S^2$ onto the plane $\mathbb{C}$. In this projection, $\infty$ is mapped to the north pole, and convergence in $\mathbb{C}_\infty$ is equivalent to convergence in $S^2$, viewed as a subset of $\mathbb{R}^3$.