Complex differentiation

- **Complex differentiable.** A complex function $f : U \to \mathbb{C}$, with $U \subset \mathbb{C}$ open, is *complex (or* $\mathbb{C}$*-)differentiable* at the point $z_0 \in U$ if the limit

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists; this limit is called the *derivative* $f'(z_0)$ of f at z. Since this is *formally* identical to the definition of *real* differentiability, all the standard computational rules of derivatives (linearity, product, quotient, and chain rules) apply with *identical proofs*.

- **Real differentiability.** A mapping $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is *real (or* $\mathbb{R}$*-)differentiable* at the vector $\mathbf{z}_0$ if there exists a real-valued matrix D such that

$$\lim_{\mathbf{h}\to 0} \frac{\left|\left|f(\mathbf{z}_0+\mathbf{h})-f(\mathbf{z}_0)-D\,\mathbf{h}\right|\right|}{\left|\left|\mathbf{h}\right|\right|} = 0;$$

such a matrix is called the *Jacobian matrix* $J_f(\mathbf{x}_0)$ of f at $\mathbf{z}_0$, and it is given by the matrix of partial derivatives; i.e. if $f = (u, v)$, then

$$J_f(\mathbf{z}_0) = \begin{pmatrix} u_x(\mathbf{z}_0) & u_y(\mathbf{z}_0) \\ v_x(\mathbf{z}_0) & v_y(\mathbf{z}_0) \end{pmatrix}.$$

- **Continuous differentiability.** A function $f = U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is called *continuously differentiable*, or of *class* C^1, if both partial derivatives f_x and f_y exists and are continuous. From *Calculus III* we have one-way implications:

$$[f \text{ is } C^1 \text{ in } U] \Longrightarrow [f \text{ is } \mathbb{R}\text{-differentiable in } U] \Longrightarrow [f \text{ has partial derivatives in } U]$$

- **Cauchy-Riemann equations.** Given a complex function $f = u + i\,v$, the *Cauchy-Riemann (CR) equations* are:

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z), \qquad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z).$$

Notice that f satisfies the CR equations iff its Jacobian matrix is an amplitwist matrix.

- **The CR condition.** A complex function f is $\mathbb{C}$-differentiable iff f is $\mathbb{R}$-differentiable and satisfies the CR equations. Moreover,

$$f'(z) = \frac{\partial f}{\partial x}(z) = \frac{\partial u}{\partial x}(z) + i\,\frac{\partial v}{\partial x}(z), \qquad J_f(z) = \begin{pmatrix} \uparrow & \uparrow \\ f'(z) & f'(z)i \\ \downarrow & \downarrow \end{pmatrix}$$

In effect, we can think of *CR* as also standing for *Complex-Real*, since it is the criterion for equivalence.

Holomorphy

- **Real and complex substitutions.** Any complex function can be viewed as a vector-valued function of the (real) variable (x, y) under the substitutions

$$z = x + i\,y, \qquad \overline{z} = x - i\,y$$

and expanding out; this is called the *real substitution* for f, and we write $f(z) = f(x, y)$. Similarly, any complex function can be viewed as a complex-valued function of the (complex) variables $(z, \overline{z})$ under the substitutions

$$x = \operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad y = \operatorname{Im} z = \frac{z - \overline{z}}{2i};$$

this is called the *complex substitution* for f, and we write $f(z) = f(z, \overline{z})$.

- **Holomorphy.** A C^1 function $f(z) = f(z, \overline{z})$ is called

$$\textit{holomorphic} \text{ at } z_0 \text{ if } \frac{\partial f}{\partial \overline{z}}(z_0, \overline{z_0}) = 0; \qquad \textit{antiholomorphic} \text{ at } z_0 \text{ if } \frac{\partial f}{\partial z}(z_0, \overline{z_0}) = 0.$$

Essentially, holomorphic functions are those which can be written without $\overline{z}$ terms, whereas antiholomorphic functions can be written without z terms.

- **Holomorphy implies differentiability.** Using the chain rule, the *holomorphic* and *antiholomorphic* differentiation operators above are equivalent to

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It follows that a function is holomorphic iff it is $\mathbb{C}$-differentiable and C^1. Moreover,

$$f'(z) = \frac{\partial f}{\partial z}(z).$$

- **Immediate consequences.** Given a complex function $f : U \to \mathbb{C}$, we have the following straightforward relationships between these definitions.

Real condition on U		**CR condition on U**		**Complex condition on U**
[f is class C^1	+	$\frac{\partial f}{\partial \overline{z}} := \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \equiv 0$]	$\Longleftrightarrow:$	f is holomorphic
$\Downarrow \quad \not\Uparrow$		$\Updownarrow$		$\Downarrow$
[{ f is real differentiable }	+	{ J_f is an ampli-twist matrix }]	$\Longleftrightarrow$	{ f is complex differentiable }
$\Downarrow \quad \not\Uparrow$		$\Updownarrow$		$\Downarrow \quad \not\Uparrow$
[{ FALSE POSITIVE! (partial derivs exist) }	+	{ f satisfies CR equations }]	$\Longleftrightarrow$	{ FALSE POSITIVE! (weakly holomorphic) }