Derivatives and integrals over curves

- **Infinitesimal amplitwists.** If f is $\mathbb{C}$-differentiable at z_0, then f acts locally near $f(z_0)$ as a multiplication by $f'(z_0)$. Said differently, close to z_0 and $f(z_0)$, the function f acts as an amplitwist through $f'(z_0)$: the plane rotates about $f(z_0)$ through an angle of $\arg f'(z_0)$ with a scaling factor of $|f'(z_0)|$.

- **Tangents to curves.** A curve $\sigma = (x, y) : [a, b] \to \mathbb{C}$ is differentiable at t_0 if each real-valued component function $x(t)$, $y(t)$ is differentiable at t_0. In this case, the derivative

$$\frac{d\sigma}{dt}(t_0) = \frac{dx}{dt}(t_0) + i\frac{dy}{dt}(t_0) \qquad \text{or} \qquad \sigma'(t_0) = x'(t_0) + i\, y'(t_0)$$

  is called the *tangent vector* to σ at t_0, and is graphically drawn at the point $\sigma(t_0)$.

- **Curves and differentiable functions.** If $\sigma : [a, b] \to U$ is an arc and $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then the derivative of the composite arc $\gamma := f \circ \sigma$ is

$$\frac{d\gamma'}{dt}(t_0) = \frac{\partial f}{\partial z}\big(\sigma(t_0)\big)\frac{d\sigma}{dt}(t_0) \qquad \text{or} \qquad \gamma'(t_0) = f'\big(\sigma(t_0)\big)\, \sigma'(t_0).$$

- **Conformality.** A function is called *angle-preserving* (or *isogonal*) at z if, whenever two curves γ and σ meet at an angle θ (i.e. their tangent vectors meet at an angle of θ) at z, then the composite curves $f \circ \gamma$ and $f \circ \sigma$ meet at the same angle θ at $f(z)$. It is called *conformal* if it additionally preserves the orientation of the tangents, and *anticonformal* if it reverses them.

- **$\mathbb{C}$-differentiability versus conformality.** If a complex function f is $\mathbb{C}$-differentiable at z_0 with $f'(z_0) \neq 0$, then f is conformal at z_0. Conversely, if f is both $\mathbb{R}$-differentiable and conformal at z_0, then it is $\mathbb{C}$-differentiable at z_0.

- **(Anti)Holomorphy versus (anti)conformality.** A C^1 function f is holomorphic at z_0 with $\frac{\partial f}{\partial z}(z_0) \neq 0$ if and only if f is conformal at z_0. Similarly, a C^1 function f is antiholomorphic at z_0 with $\frac{\partial f}{\partial \bar{z}}(z_0) \neq 0$ if and only if f is anticonformal at z_0.

- **Line integrals.** Let $f : U \to \mathbb{C}$ be a complex function and γ a smooth curve from α to β. The *complex line integral* of f over γ is the limit

$$\int_\gamma f\, dz := \lim_{\lambda \to 0} \sum_{k=0}^{n} f(\zeta_k)\Delta z_k,$$

  provided the limit exists. Here $\alpha = z_0, z_1, z_2, \ldots, z_n = \beta$ are points of γ arranged in the positive order, ζ_k is a point of γ on the arc between z_{k-1} and z_k, $\Delta z_k = z_k - z_{k-1}$, and λ is the maximum length of the n subarcs.

Complex line integrals

- **Properties of line integrals.** If f is continuous, then
$$\int_\gamma f\,dz = \int_\gamma (u+i\,v)(dx+i\,dy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy,$$
which is the (complex) sum of two *real* line integrals over γ. Hence, the complex line integral has the same basic properties as the real line integral: linearity, independence of parametrization, dependence on orientation, etc. An important inequality for the modulus of an integral also extends:
$$\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,|dz| =: \int_\gamma |f|\,ds \le \max_{z\in\gamma}|f(z)| \times \text{length}(\gamma).$$

- **Smooth curves.** An arc γ is called *smooth* if it can be parametrized by a path $z(t) : [a,b] \to \mathbb{C}$ with $z'(t) \neq 0$ for any $t \in (a,b)$. If a smooth arc γ has a parametrization $z = z(t) : [a,b] \to \mathbb{C}$, then
$$\int_\gamma f\,dz = \int_a^b f\big(z(t)\big)\,z'(t)\,dt.$$

- **Complex FTC (Part I):** If f has a complex antiderivative F defined an open set U. If γ is any arc in U from α to β, then
$$\int_\gamma f\,dz = F(\beta) - F(\alpha).$$
In particular, the actual arc γ is immaterial — only the endpoints matter. As a consequence, if f has different line integrals over two different paths γ and η connecting the same two points, then f cannot have an antiderivative on any neighborhood of $\gamma\cup\eta$.

- **Conservative functions.** A complex function $f : U \to \mathbb{C}$ is called *conservative* on U if
$$\int_\gamma f\,dz = 0$$
for every closed curve γ in U.

- **Green's Theorem:** This is a Calculus III result, which states that if $(P,Q) : U \to \mathbb{R}^2$ is a C^1 vector field and γ is a Jordan curve in U, then
$$\int_\gamma P\,dx + Q\,dy = \iint_{\textbf{inside}(\gamma)} \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} dx\,dy.$$

- **Cauchy's Integral Theorem:** If U is simply connected and $f : U \to \mathbb{C}$ is holomorphic, then
$$\int_\gamma f\,dz = 0$$
for every Jordan curve γ in U.