The complex elementary functions

- **The complex exponential.** The *complex exponential* is defined by

$$e^z := e^{\mathrm{Re}\, z}\, e^{i\, \mathrm{Im}\, z} = e^{\mathrm{Re}\, z}\big(\cos(\mathrm{Im}\, z) + i \sin(\mathrm{Im}\, z)\big).$$

This function is holomorphic on $\mathbb{C}$ and satisfies the properties

$$\frac{d}{dz}\{e^z\} = e^z, \qquad \big(e^z\big)\big(e^w\big) = e^{z+w}, \qquad e^{z+2\pi i} = e^z.$$

In particular, the complex exponential is periodic with imaginary period $2\pi\, i$.

- **The complex logarithm.** A *complex logarithm* is defined to be any inverse of the exponential e^z, i.e. any function $w = \log(z)$ which satisfies $e^w = z$. Since e^z is periodic, there are *infinitely many* ways to define a logarithm which differ by multiplies of $2\pi\, i$, which is precisely the period of e^z. However, any logarithm takes the form

$$\log(z) = \ln|z| + i\, \arg(z) + i(2\pi k_z), \qquad k_z \in \mathbb{Z}.$$

If we have a logarithm $\log(z)$ define on an open set U, then it is called a *branch* of the logarithm if it is continuous on U. In fact, any branch of the logarithm is holomorphic with derivative

$$\frac{d}{dz}\big\{ \log(z)\big\} = \frac{1}{z}.$$

Observe that the domain U of a branch of the logarithm cannot contain 0 (since $e^z \neq 0$ for any z); moreover, a maximal domain consists of a *branch cut* of points (starting at 0) in $\mathbb{C}$ removed from the domain of definition.

- **The principal branch of the logarithm.** If we restrict the argument to the principal argument, then we can define the *principal branch* of the logarithm by

$$\mathrm{Log}(z) := \ln|z| + i\, \arg_p(z), \qquad -\pi < \arg_p(z) < \pi.$$

Notice that $\mathrm{Log}(z)$ is not continuous across its branch cut line $\{x \leq 0\}$. For example, if we approach the number $z = -1$ from Quadrant II, then $\mathrm{Log}(z) \to i\pi$; however, if we approach from Quadrant III, then $\mathrm{Log}(z) \to -i\pi$. This kind of behavior, having the logarithm differ by a multiple of $2\pi\, i$ on either side of the branch cut, is common to all logarithms. A consequence of this is that, in general,

$$\log(z\, w) \neq \log(z) + \log(w), \qquad \log(z^n) \neq n\, \log(z).$$

- **The complex sine and cosine.** We define the two basic *complex trigonometric* functions as follows

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

This implies that sine and cosine are holomorphic functions on $\mathbb{C}$, and satisfies the usual properties

$$\frac{d}{dz}\big\{ \cos(z)\big\} = -\sin(z), \qquad \frac{d}{dz}\big\{ \sin(z)\big\} = \cos(z), \qquad \sin^2(z) + \cos^2(z) = 1.$$

Cauchy's Theorem and its consequences

- **Where we left off...** On Friday, we used Green's Theorem to prove the following: if $U \subset \mathbb{C}$ is a simply connected domain and $f : U \to \mathbb{C}$ is *holomorphic*, then

$$\int_\gamma f\, dz = 0$$

for any *Jordan curve* γ in U. It turns out that the hypotheses that f be C^1 and that γ be a non-self-intersecting Jordan curve can be eliminated:

- **Cauchy's Integral Theorem.** If $U \subset \mathbb{C}$ is a simply connected domain and $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then

$$\int_\gamma f\, dz = 0$$

for any closed curve γ in U, i.e. f is conservative on U.

- **Application 1: Complex FTC (Part II).** If f is $\mathbb{C}$-differentiable on a simply connected domain U and $z_0 \in U$ is arbitrary, then

$$F(z) := \int\limits_{\gamma[z_0,z]} f(\zeta)\, d\zeta,$$

where $\gamma[z_0, z]$ denotes *any* path in U beginning at z_0 and ending at z, defines a $\mathbb{C}$-differentiable function with derivative $f(z)$. Moreover, if $\tilde{F}$ is another antiderivative, then

$$\tilde{F}(z) \equiv F(z) + C$$

for some complex constant C. Note that, unlike the real case, the mere continuity of f is insufficient to guarantee that a *complex* antiderivative.

- **Logarithms as antiderivatives.** The set $U = \mathbb{C} \setminus \{x \leq 0\}$ is simply connected, and $1/z$ is holomorphic in it. Hence, the FTC guarantees that $1/z$ has an antiderivative on it. In fact, the usual antiderivative is the principal branch of the logarithm:

$$\mathrm{Log}(z) = \int\limits_{\gamma[1,z]\subset U} \frac{1}{\zeta}\, d\zeta, \quad z \in U.$$

- **Application 2: Homotopy version.** Suppose that γ and η are Jordan curves, both oriented positively, and suppose that $\eta \subset \mathrm{inside}(\gamma)$. Let R be the region "between" the curves, i.e. R is the closure of the domain $\mathrm{inside}(\gamma) \cap \mathrm{outside}(\eta)$. If a complex function f is $\mathbb{C}$-differentiable on a neighborhood of R, i.e. $\mathbb{C}$-differentiable on the "in-between" domain and at every point on either curve γ or η, then

$$\int_\gamma f\, dz = \int_\eta f\, dz.$$

This allows us to exchange complicated closed paths for simpler closed paths when they enclose "bad" points of a function. This theorem also applies to the case when there are several disjoint, positively-oriented Jordan curves inside γ.