The Integral Theorems of complex analysis

- **The proof of Cauchy's Integral Theorem.** The proof involves two steps:
  - *Step 1. The Cauchy-Goursat Theorem*: If U is simply connected and $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable on U, then
  $$\int_{\Delta} f\,dz = 0$$
  for every triangular path Δ in U. A proof is given in the class handout for today.
  - *Step 2. Polygonal approximation*: Any line integral can be approximated within an ϵ by a line integral over a polygonal path.

  Since Step 1 implies that every integral over a *closed* polygonal path is 0, the result follows from Step 2.

- **Application 3: Cauchy's Integral Formula.** Let U be simply connected and $f : U \to \mathbb{C}$ be $\mathbb{C}$-differentiable. If γ is a positively-oriented Jordan curve in U around the point z^*, then
$$f(z^*) = \frac{1}{2\pi\, i} \int_{\gamma} \frac{f(z)}{z - z^*}\,dz.$$
Its proof, similar to that of the Cauchy-Goursat Theorem, involves integrating the continuity condition.

- **Boundary uniqueness.** A $\mathbb{C}$-differentiable function is determined by its values on the boundary. Suppose that $f, g : U \to \mathbb{C}$ are $\mathbb{C}$-differentiable and $f(z) = g(z)$ for all z in a Jordan curve γ in U, then in fact
$$f(z) \equiv g(z) \quad \forall\, z \in \text{inside}(\gamma).$$

- **Application 4: $\mathbb{C}$-Differentiability and the CIF.** Let U be simply connected. A continuous function $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable if and only if it satisfies the Cauchy Integral Formula. Moreover, if γ is a positively-oriented Jordan curve in U around the point z^*, then
$$f'(z^*) = \frac{1}{2\pi\, i} \int_{\gamma} \frac{f(z)}{(z - z^*)^2}\,dz.$$

- **Infinite differentiability.** A mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is called *infinitely differentiable*, or *smooth*, or of class C^∞, if it has continuous partial derivatives of all orders. Clearly, C^∞ implies C^1, but the converse is not, in general, true.

- **$\mathbb{C}$-differentiability implies smoothness.** If U is simply connected and $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable, then f is smooth on U. Moreover, if γ is any Jordan curve around z^*, then
$$f^{(n)}(z^*) = \frac{n!}{2\pi\, i} \int_{\gamma} \frac{f(z^*)}{(z - z^*)^{n+1}}\,dz.$$

Equivalences

Given a complex function $f : U \to \mathbb{C}$,

$\mathbb{R}$-cond. on U		CR cond.		$\mathbb{C}$-cond on U		Simply connected?
[f is class C^∞	+	CR]	$\Longleftrightarrow$	f is smoothly holomorphic	$\Longleftrightarrow$	f satifies C.I.F.
$\Downarrow$ $\not\Uparrow$		$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
[f is class C^1	+	CR]	$\Longleftrightarrow:$	f is holomorphic	$\Longleftrightarrow$	f is conservative
$\Downarrow$ $\not\Uparrow$		$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
[f is real differentiable	+	CR]	$\Longleftrightarrow$	f is complex differentiable	$\Longleftrightarrow$	f has an antiderivative
$\Downarrow$ $\not\Uparrow$		$\Updownarrow$		$\Updownarrow$ $\not\Uparrow$		
[partial derivs exist	+	CR]	$\Longleftrightarrow$	f is weakly holomorphic		[**FALSE POSITIVE!**]

Using the integral theorems

- **Closed curves.** Suppose you wish to integrate a holomorphic function f over a closed curve γ. First, break down γ into a sum of smaller Jordan curves and integrate over each Jordan curve separately. Then:
  - *Antiderivative?* If f has an obvious antiderivative on any neighborhood of γ, simply connected or not, then the FTC implies $\int_\gamma f\,dz = 0$.
  - *Simply connected?* If f is holomorphic on a *simply-connected* neighborhood of γ, then the CIT implies $\int_\gamma f\,dz = 0$.
  - *Lots of bad points?* Use the homotopy version of CIT to write this as several smaller circles, each about a single bad point.
  - *One bad point?* Write f as $g(z)/(z - z^*)^{k+1}$ with g holomorphic in the curve. Then the CIF implies $\int_\gamma f\,dz = \frac{2\pi i}{k!} g^{(k)}(z^*)$.
  - *If all else fails...* Parametrize the curve and do it by hand.
- **Arcs.** Suppose you wish to integrate a holomorphic function f over a *non-closed* arc γ. Then:
  - *Antiderivative?* If f has an antiderivative F on any neighborhood of γ, simply connected or not, then the FTC implies $\int_\gamma f\,dz = F(\mathbf{end}) - F(\mathbf{start})$.
  - *Easier path to parametrize?* Parametrize an easier path η going between the same points calculate $\int_\eta f\,dz$. Then since $\gamma - \eta$ is closed, you can use the techniques above the calculate the integral over the closed loop. The subtract the integrals.
  - *If all else fails...* Parametrize the curve and do it by hand.