Power functions

- **Complex power functions.** For q a complex number, we define the q-th power function to be

$$z^q := e^{q\,\log(z)},$$

where log denotes a branch of the logarithm. Since this defines z^q as a multifunction, there are *many different* ways to define a q-th power which differ by a factor of $e^{2\pi q\,i}$. Hence, any q-th power takes the form

$$z^q = e^{q\ln(|z|)+i\,q\,\arg(z)} \cdot \left(e^{2\pi q\,i}\right)^{k_z}, \qquad k_z \in \mathbb{Z}.$$

If we have a q-th power function z^q defined on an open set U, then it is called a *branch* of the q-th power if it is continuous on U. In fact, any branch of a q-th power is holomorphic with derivative

$$\frac{d}{dz}\left\{z^q\right\} = \frac{q\,z^q}{z}.$$

Observe that the domain U of a branch of a root cannot contain 0, since every neighborhood of 0 has multiple preimages; moreover, a maximal domain consists of a *branch cut* of points (starting at 0) in $\mathbb{C}$ removed from the domain of definition.

- **The principal branch of the q-th power.** If we restrict the argument to the principal argument, then we can define the *principal branch* of the q-th power by

$${z_p}^q := e^{q\,\mathrm{Log}(z)} = e^{q\ln(|z|)+i\,q\,\arg_p(z)}, \qquad -\pi < \arg_p(z) < \pi.$$

Notice that ${z_p}^q$ is not continuous across its branch cut line $\{x \le 0\}$. A consequence of this is that, in general,

$$(z\,w)^q \neq z^q\,w^q, \qquad \left(z^q\right)^r \neq z^{qr}.$$

- **Antiderivatives.** Each simply-connected branch of a q-th power has an antiderivative given by

$$\int z^p\,dz = \frac{z\,z^p}{p}.$$

- **Consistency.** One can check that if q is an integer, then every branch of z^q coincides with the usual monic polynomial. Similarly, if $q = 1/n$, then every branch coincides with an inverse of z^n.

Circles in $\mathbb{C}$

- **Lines as circles.** We shall refer to lines as *circles through the point at infinity*, since the stereographic projection of every line in $\mathbb{C}$ onto the Riemann sphere $\mathbb{C}_\infty$ is a circle. We also refer to lines as *infinite circles*, whereas for comparison, true circles are called *finite circles*. Any circle can be written of the form

$$Az\,\overline{z} + \overline{B}\,z + B\overline{z} + C = 0,$$

with A, C real and $|B|^2 - AC > 0$.

- **Circle symmetry.** Two points z, w are *symmetric* about the finite circle $\partial B(\zeta, R)$ if (a) ζ, z and w are collinear and (b)$|z-\zeta|\,|w-\zeta| = R^2$.

  On the other hand, z, w are *symmetric* about a line ℓ if z, w are mirror images across the line, i.e. if ℓ is the perpendicular bisector of the segment $[z, w]$.

- **Circle-preserving functions.** A function is called *circle-preserving* if the image of every (possibly infinite) circle is a (possibly) infinite circle. The three most basic circle-preserving functions are *translations* $z \mapsto z+w$, *amplitwists* $z \mapsto w\,z$, and *the inversion* $z \mapsto 1/z$.

  Moreover, these functions also preserve symmetry, in the following sense: if z, w are symmetric about the circle C, then $f(z), f(w)$ are symmetric about the circle $f(C)$.

- **Möbius transformations.** A *Möbius transformation*, or *fractional linear transformation*, is a complex function of the form

$$f(z) := \frac{a\,z+b}{c\,z+d}.$$

  Observe that every Möbius transformation can be written as

$$f(z) = \frac{a}{c} + \frac{bc-ad}{c}\frac{1}{c\,z+d},$$

  and hence is a composition of translations, amplitwists, and inversions. Thus, Möbius transformations preserve circles and circle symmetry.

- **Bijections of the sphere.** Note that for a Möbius transformation,

$$f(\infty) = \frac{a}{c}, \qquad f\left(-\frac{d}{c}\right) = \infty.$$

  Hence, Möbius transformations are continuous, one-to-one mappings of the Riemann sphere $\mathbb{C}_\infty$ onto itself. Moreover, given any assignment of values to three points

$$z_1 \mapsto w_1, \qquad z_2 \mapsto w_2, \qquad z_3 \mapsto w_3,$$

  there is a *unique* Möbius transformation satisfying these three conditions.