Complex series

- **Series.** A *complex series* is a formal infinite sum of complex numbers

$$\sum_{n=0}^{\infty} a_n \equiv \sum_{n\in\mathbb{Z}^+} a_n := a_0 + a_1 + a_2 + \cdots + a_n + \cdots .$$

To each series is a corresponding sequence (S_n) of partial sums, i.e.

$$S_k := \sum_{n=0}^{k} a_n = a_0 + a_1 + a_2 + \cdots + a_{k-1} + a_k.$$

A series *converges* to the sum α if the sequence S_n converges to α, and this is denoted $\sum a_n = \alpha$. A series which does not converge is said to *diverge*.

- **Absolute convergence.** A series $\sum a_n$ converges *absolutely* if the real series $\sum |a_n|$ converges. Absolute convergence implies convergence but not conversely; the alternating harmonic series is a counterexample. Such series are said to converge *conditionally*.

- **Manipulating series.** Series can be manipulated freely *provided* they converge absolutely. For example, if $\sum a_n$ and $\sum b_n$ converge absolutely, then

$$\begin{aligned} \textit{Linear combos:}\quad & \zeta \sum_{n=0}^{\infty} a_n + \xi \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (\zeta\, a_n + \xi\, b_n) \\ \textit{Products:}\quad & \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ \textit{Rearrangements:}\quad & \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\sigma(n)} \quad \forall \text{ permutation } \sigma \text{ of } \mathbb{Z}^+ \end{aligned}$$

However, these are not, in general, true of conditionally convergent series.

- **Geometric series.** The geometric series is defined by

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + a^4 + \cdots .$$

This converges absolutely if $|a| < 1$ and diverges otherwise. In fact,

$$S_k = \sum_{n=0}^{k} a^n = \frac{1 - a^{k+1}}{1 - a}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} a^n = \begin{cases} \dfrac{1}{1-a} & \text{if } |a| < 1 \\ \text{diverges} & \text{if } |a| \geq 1 \end{cases}$$

- **Harmonic series.** The harmonic series is defined by

$$\sum_{n=0}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

This is a famously *divergent* series; indeed, if S_n is the n-th partial sum, then $S_{2^n} \geq \frac{n}{2}$.

Convergence Tests

- **Divergence test.** If the terms $a_n \not\to 0$ as $n \to \infty$, then $\sum a_n$ diverges. The converse statement — if the terms $a_n \to 0$ as $n \to \infty$, then $\sum a_n$ converges — is *not true*; the harmonic series is a counterexample.

- **Comparison Test.** Given a series $\sum a_n$, if

$$0 \leq |a_n| < r_n \text{ and } \sum_{n \in \mathbb{Z}^+} r_n \text{ converges} \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ converges absolutely}$$
$$|a_n| > r_n \geq 0 \text{ and } \sum_{n \in \mathbb{Z}^+} r_n \text{ diverges} \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ diverges}$$

- **The Ratio Test.** Given a series $\sum a_n$, set

$$L := \overline{\lim_{n \to \infty}} \left| \frac{a_{n+1}}{a_n} \right|,$$

  where $\overline{\lim}$ denotes the *lim sup*, i.e. the supremum of the set of limit points of the sequence. Then

$$\text{If } L < 1 \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ converges absolutely}$$
$$\text{If } L > 1 \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ diverges}$$
$$\text{If } L = 1 \Longrightarrow \text{no information about convergence}$$

- **The Root Test.** Given a series $\sum a_n$, set

$$L := \overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|},$$

  Then

$$\text{If } L < 1 \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ converges absolutely}$$
$$\text{If } L > 1 \Longrightarrow \sum_{n \in \mathbb{Z}^+} a_n \text{ diverges}$$
$$\text{If } L = 1 \Longrightarrow \text{no information about convergence}$$

- Both the Ratio and Roots Tests are obtained by comparing the series $\sum a_n$ to a suitable geometric series, the workhorse of all series.