Series of functions

- **Series of functions.** Suppose that $f_n(z)$ is a complex function for $n = 0, 1, 2, \ldots$, and the define the function f by

$$f(z) := \sum_{j=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + f_2(z) + \cdots .$$

The function f *converges (pointwise)* on U if the series of complex numbers $\sum f_n(z)$ converges for each $z \in U$, i.e.

$$\forall \epsilon > 0,\ z_0 \in U\, \exists K = K(\epsilon, z_0) \geq 1 \quad \text{s.t.} \quad k \geq K, \Longrightarrow \left| f(z_0) - \sum_{n=0}^{k} f_n(z_0) \right| < \epsilon.$$

Observe that in the ϵ-K definition of convergence, the choice of K depends on both ϵ and the specific point z_0 in question. Lastly, f converges *absolutely* on U if the series $\sum \left|f_n(z)\right|$ converges.

- **Uniform convergence.** The sum f converges *uniformly* on U if

$$\forall \epsilon > 0\, \exists K = K(\epsilon) \geq 1 \quad \text{s.t.} \quad k \geq K,\ z \in U \Longrightarrow \left| f(z) - \sum_{n=0}^{k} f_n(z) \right| < \epsilon.$$

The point is that one single value of K will work for every value of z in U. This condition was formulated by Weierstrass to fix the proof of Cauchy's False Theorem.

- **Weierstrass M-test.** Suppose $f_n : U \to \mathbb{C}$ are complex functions and

$$\sup_{z \in U} \left|f_n(z)\right| \leq M_n, \qquad \sum_{j=0}^{\infty} M_n < \infty,$$

then $\sum f_n$ converges uniformly and absolutely on U.

- **Continuity.** Suppose $\sum f_n$ converges uniformly to f on U. If each function $f_n(z)$ is continuous at z_0, then $f(z)$ is continuous at z_0.

- **Integrability.** Suppose $\sum f_n$ converges uniformly to f on a neighborhood U of a curve γ. Then

$$\int_\gamma f(z)\, dz = \sum_{n=0}^{\infty} \int_\gamma f_n(z).$$

Notice that neither U need be simply connected not γ be closed.

- **Differentiability.** Suppose $\sum f_n$ converges uniformly to f on an *open ball* B. If each f_n is $\mathbb{C}$-differentiable on B, then f is $\mathbb{C}$-differentiable on B as well. Moreover,

$$f^{(k)}(z) = \sum_{n=0}^{\infty} {f_n}^{(k)}(z).$$

This theorem has no trivial analog in the real case: many more strict hypotheses are required before the limit of real derivatives is the derivative of the limit.

Power series and analyticity

- **Power series.** A *power series* is a sum of the form

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots .$$

The value a is called the *center* of the power series. Notice that every power series converges at its center. Moreover, we have...

- **Abel's Lemma.** Suppose a power series $f(z) = \sum c_n (z-a)^n$ converges for some $\zeta \neq a$. Then $f(z)$ converges absolutely on the open disk $B\big(a, |\zeta - a|\big)$ and uniformly on every proper subdisk.

- **Radius of convergence.** The *radius of convergence* is the value $R \geq 0$ such that the series $f(z) = \sum c_n (z-a)^n$ converges absolutely for all $|z-a| < R$ and diverges for all $|z-a| > R$. Moreover, the $f(z)$ convergence is uniform on any proper subdisk of the maximal one.

- **Hadamard's Theorem.** The radius of convergence R of $\sum c_n (z-a)^n$ can be found by

$$\frac{1}{R} = \overline{\lim_{n \to \infty}} \sqrt[n]{|c_n|} \quad \text{where } \frac{1}{0} = \infty, \ \frac{1}{\infty} = 0.$$

- **Analytic functions.** A function $f : U \to \mathbb{C}$ is *analytic* at $a \in U$ if f can be expressed locally at a as a power series, i.e. there exists a power series $\sum c_n (z-a)^n$ centered at a with a positive radius of convergence R such that

$$f(z) \equiv \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad \forall \, |z-a| < R;$$

Such an power series is called a *power series expansion* of f at a. f is *analytic in U* if it is analytic at each point $a \in U$.

- **Analyticity implies $\mathbb{C}$-differentiability.** If f is analytic at a, then f is $\mathbb{C}$-differentiable on a neighborhood of a, and term-by-term differentiation (and integration) is permissable on the neighborhood, i.e.

$$f'(z) = \sum_{n=1}^{\infty} n \, c_n (z-a)^{n-1}, \qquad \int f(z)\, dz = K + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (z-a)^{n+1}.$$

- **Taylor coefficients.** As a consequence of holomorphy, the coefficients c_n are uniquely determined by

$$c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = 2\, f''(a), \quad c_3 = 6\, f'''(a), \quad \ldots, \quad c_n = n!\, f^{(n)}(a).$$

and are called the *Taylor coefficients* of f at a. By the Cauchy Integral Formula, we can also write

$$c_n = \frac{1}{2\pi\, i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - a)^{n+1}} \, d\zeta$$

where γ is any positively-oriented Jordan curve around a.

Taylor series expansions

- **Taylor series.** Given any $\mathbb{C}$-differentiable function f, its *Taylor series* at a is the power series defined by setting $c_n = n!\, f^{(n)}(a)$, i.e. the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots .$$

  Observe that the power series expansion of any analytic function is precisely its Taylor series.

- **$\mathbb{C}$-differentiability implies analyticity.** Suppose that $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable, and suppose that $B(a,R) \subset U$. Then f is the sum of its Taylor series, and its radius of convergence is at least R.

- **Taylor series revisited.** The coefficients c_n of the Taylor series $\sum c_n(z-a)^n$ of a $\mathbb{C}$-differentiable function f at a point a can also be expressed in terms of integrals, namely

$$c_n = \frac{1}{2\pi\, i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \qquad n = 0, 1, 2, \ldots$$

  where γ is any Jordan curve around the point a.

- **Consequences of equivalence.** Any power series expansion of a function agrees with its Taylor series expansion, so a series expansion can be determined either by manipulating a known series or by taking derivatives and plugging into the Taylor formula. Similarly, the radius of convergence can be determined either by Hadamard's Theorem (in series form) or as the distance to the nearest "bad point" (in function form).