Uniqueness Theorems

- **The Zero Lemma.** Let $f : U \to \mathbb{C}$ be holomorphic and U be open. If the set of points

$$\{z \in U : f(z) = 0\}$$

has a limit point $z^* \in U$, then $f(z) \equiv 0$ on any neighborhood of z^*. This is a power series theorem, which has no analog for smooth real functions.

- **The Uniqueness Theorem.** Suppose U is a domain (i.e. open and connected) and $f, g : U \to \mathbb{C}$ are holomorphic. If the set

$$\{z \in U : f(z) = g(z)\}$$

has a limit point in U, then $f(z) \equiv g(z)$ for every $z \in U$. Said differently, a holomorphic function is *uniquely determined* by its values on any convergent sequence of distinct points.

- **The Mean Value Property.** A function $f : U \to \mathbb{C}$ is said to have the *Mean Value Property (MVP)* if the value at any point is the mean value over any circle centered at that point, i.e. if

$$f(z^*) = \frac{1}{2\pi} \int_0^{2\pi} f\big(z^* + R\, e^{i\theta}\big)\, d\theta$$

for any point z^* and any radius R such that $\overline{B(z^*, R)} \subset U$.

For complex functions, this is equivalent to satisfying the Cauchy Integral Formula on circles only. Hence, every holomorphic function satisfies the Mean Value Property.

- **Maximum Modulus Principle.** A non-constant function on a domain cannot assume a maximum modulus. More precisely, suppose that $f : U \to \mathbb{C}$ is holomorphic with U a domain. If there exists a point $z^* \in U$ such that

$$\big|f(z)\big| \le \big|f(z^*)\big| \quad \forall\, z \in U,$$

then $f(z) \equiv f(z^*)$ for every point $z \in U$.

- **Consequences.** The Maximum Modulus Principle has several useful corollaries:
  - *Boundary Value Principle.* If U is a bounded domain, and f is continuous on $\overline{U}$ and holomorphic in U, then f assumes its maximum modulus on the boundary of U.
  - *Boundary determination.* Suppose γ is a Jordan curve and f, g are continuous *on and inside* γ. If $f = g$ on the curve γ, and f, g are holomorphic inside γ, then $f \equiv g$ inside γ.
  - *Minimum Modulus Principle.* If U is a domain and $f : U \to \mathbb{C}$ is holomorphic, non-constant, and *nonzero* on U, then f does not attain a minimum modulus on U.

Laurent Series

- **Singular series.** A *singular series* is a sum of the form

$$f(z) = \sum_{n=0}^{\infty} c_{-n}(z-a)^{-n} = c_0 + \frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \cdots .$$

  The value a is called the *center* of the singular series. A singular series centered at a is, essentially, nothing more than a power series in the variable $(z-a)^{-1}$. Hence, for each power series result there is a corresponding singular series result formed by "inversion.".

- **Radius of divergence.** The *radius of divergence* is the value $R_d \geq 0$ such that the singular series $f(z) = \sum c_{-n}(z-a)^{-n}$ diverges for all $|z-a| < R_d$ and converges absolutely for all $|z-a| > R_d$. Moreover, the convergence is uniform on for $|z-a| \geq R > R_d$. The radius of divergence R_d of $\sum c_{-n}(z-a)^{-n}$ can be found by

$$R_d = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_{-n}|} \quad \text{or} \quad R_d = \overline{\lim_{n\to\infty}} \left| \frac{c_{-(n+1)}}{c_{-n}} \right| .$$

- **Laurent series.** A *Laurent series* is a doubly-infinite sum of the form

$$f(z) \equiv \sum_{n\in\mathbb{Z}} c_n(z-a)^n = \sum_{n=-\infty}^{\infty} c_n(z-a)^n := \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n} + \sum_{n=0}^{\infty} c_n(z-a)^n$$

$$= \underbrace{\cdots + \frac{c_{-3}}{(z-a)^3} + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a}}_{\text{singular part}} + \underbrace{c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots}_{\text{regular part}}$$

  The value a is called the *center* of the Laurent series.

  Notice that a Laurent series is just the sum of two series of functions: a singular series at (with no constant term), called its *singular part*, and a power series, called its *regular part*.

- **Annulus of convergence.** If R_c denotes the radius of convergence of the regular part and R_d the radius of divergence of the singular part, then the Laurent series $\sum_{n\in\mathbb{Z}} c_n(z-a)^n$ converges absolutely on its *annulus of convergence*,

$$A(a; R_d, R_c) := \{z \in \mathbb{C} : R_d < |z-a| < R_c\},$$

  and diverges in the exterior of the annulus. Moreover, the convergence is uniform in any proper sub-annulus $A(a; r, R)$ with $R_d < r < R < R_c$.

- **Differentiability.** A Laurent series $f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$ is differentiable (and integrable) in its annulus of convergence, and term-by-term operations are permissible.

- **Laurent coefficients.** *Laurent coefficients* of the series are uniquely determined by its *annulus of convergence* according to the formula

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, dz, \qquad n = 0, \pm1, \pm2, \pm3, \ldots,$$

  where γ is any Jordan curve contained in the annulus which wraps around a.