Laurent Expansion

- **Laurent Expansion Theorem.** Suppose that f is holomorphic on an open annulus $A(a;r,R)$. Then f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n\,(z-a)^n$$

which converges absolutely in the annulus and uniformly on compact subannuli. Moreover, the coefficients c_n of the Laurent expansion are determined uniquely as

$$c_n = \frac{1}{2\pi\, i}\int_\gamma \frac{f(\zeta)}{\zeta - a}\,d\zeta, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

where γ is any positively oriented Jordan curve in the annulus which wraps around a.

- **Local Laurent expansion.** A Laurent series expansion of f on a "deleted neighborhood" of a, i.e. an annulus of the form $A(a;0,R)$, is called the *local* Laurent expansion of f at a.

- **Laurent estimates.** These are analogs of the Cauchy Estimates for holomorphic functions. If $f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n$ is a Laurent expansion in the annulus $A(a;R_1,R_2)$, then for every $R_1 < R < R_2$, we have

$$|c_n| \le \frac{M_R}{R^n}, \quad \text{where } M_R := \max_{|z-a|=R} \big|f(z)\big|.$$

- **Useful Laurent series.** In practice, its useful to know a couple of important Laurent (and Taylor) series. Among them

Geometric $(\lvert z\rvert < 1)$	$\dfrac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$
Geometric $(\lvert z\rvert > 1)$	$\dfrac{1}{1-z} = \dfrac{1}{-z\big(1-\frac{1}{z}\big)} = \sum_{n=1}^{\infty} -\dfrac{1}{z^n} = -\dfrac{1}{z} - \dfrac{1}{z^2} - \dfrac{1}{z^3} - \cdots$
Derivatives!	$\dfrac{1}{(1-z)^{k+1}} = \dfrac{1}{k!}\dfrac{d^k}{dz^k}\left\{\dfrac{1}{1-z}\right\}$
Binomial $(\lvert z\rvert < 1)$	$(1+z)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n} z^n = \sum_{n=0}^{\infty} \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\, z^n$
Exponential $(\forall\, z \in \mathbb{C})$	$e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{1}{2}z^2 + \dfrac{1}{6}z^3 + \dfrac{1}{24}z^4 + \cdots$
Sine $(\forall\, z \in \mathbb{C})$	$\sin z = \sum_{n=0}^{\infty} \dfrac{z^{2n-1}}{(2n-1)!} = z - \dfrac{1}{3!}z^3 + \dfrac{1}{5!}z^5 - \dfrac{1}{7!}z^7 + \cdots$
Cosine $(\forall\, z \in \mathbb{C})$	$\cos z = \sum_{n=0}^{\infty} \dfrac{z^{2n}}{(2n)!} = 1 - \dfrac{1}{2!}z^2 + \dfrac{1}{4!}z^4 - \dfrac{1}{6!}z^6 + \cdots$

Special types of points

- **Zeros.** A *zero* of a nonconstant function f is a point a such that $f(a) = 0$. If f is holomorphic and a is a zero of f, then we can Laurent (or Taylor!) expand f as

$$f(z) = \sum_{n=m}^{\infty} c_n (z-a)^n = c_m (z-a)^m + c_{m+1}(z-a)^{m+1} + \cdots .$$

The smallest $m \geq 1$ such that $c_m \neq 0$ is called the *order* of the zero. Then the following are equivalent:

$$\begin{aligned} a \text{ is a zero of order } m \quad &\iff \quad f(a) = f'(a) = \cdots = f^{(m-1)}(0) = 0,\ f^{(m)}(0) \neq 0 \\ &\iff \quad \text{there exists a } \textit{nonzero} \text{ analytic } g \text{ defined near } a \\ &\qquad\quad \text{s.t.} \quad f(z) = (z-a)^m g(z) \quad \forall\, z \text{ near } a \end{aligned}$$

- **Isolated singularities.** A point a is an *isolated singularity* of f if f is *not* differentiable at a, but *is* differentiable on a *deleted neighborhood* of a, i.e. an annulus of the form $A(a;0,R)$. There are three types of isolated singularities classified by the behavior of f near a:

$$\begin{aligned} \textit{Removable:} \quad & \lim_{z\to a} f(z) = A \in \mathbb{C}, \text{ so } f \text{ can be made continuous at } a \\ \textit{Pole:} \quad & \lim_{z\to a} f(z) = \infty, \text{ so } f \text{ can be made continuous in } \mathbb{C}_\infty \\ \textit{Essential:} \quad & \lim_{z\to a} f(z) \text{ does not exist, so } f \text{ cannot be made continuous} \end{aligned}$$

- **Removable singularities.** The following conditions are also equivalent for an isolated singularity a of a function f:

$$\begin{aligned} a \text{ is removable} \quad &\iff \quad f \text{ can be made continuous at } a \\ &\iff \quad f \text{ is bounded on a neighborhood of } a \\ &\iff \quad \text{the local Laurent series at } a \text{ has } \textit{no} \text{ singular part} \\ &\iff \quad f \text{ can be made analytic at } a \end{aligned}$$

- **Poles.** It is easy to see that $f(z)$ has a pole at a if and only if the reciprocal function $\frac{1}{f(z)}$ has a removable zero at a. Hence, every pole has a corresponding *order*. In fact,

$$\begin{aligned} a \text{ is pole of order } m \quad &\iff \quad 1/f \text{ has a removable zero of order } m \text{ at } a \\ &\iff \quad \text{there exists a } \textit{nonzero} \text{ analytic } g \text{ defined near } a \\ &\qquad\quad \text{s.t.} \quad f(z) = \frac{g(z)}{(z-a)^m} \quad \forall\, z \text{ near } a \\ &\iff \quad \text{the local Laurent series at } a \text{ has } \textit{finite} \text{ singular part} \end{aligned}$$

- **Essential singularities.** The following are equivalent for an isolated singularity a:

$$\begin{aligned} a \text{ is an essential singularity} \quad &\iff \quad \text{the local Laurent series at } a \text{ has } \textit{infinite} \text{ singular part} \\ &\iff \quad \text{for any } A \in \mathbb{C}_\infty, \text{ there exists a sequence } z_n \\ &\qquad\quad \text{s.t.} \quad z_n \to a,\ f(z_n) \to A \quad \text{as} \quad n \to \infty \end{aligned}$$