Residues

- **Residues.** Suppose f is analytic on a deleted neighborhood of $z = a$. Then the *residue* of f at a is the complex number

$$\operatorname{Res}(f, a) := \lim_{R \to 0} \frac{1}{2\pi\, i} \int_{|z-a|=R} f(z)\, dz,$$

  where the circle is integrated counter-clockwise.

- **Residues and Laurent expansions.** Given the *local* Laurent expansion of f at a, i.e. the expansion in the annulus $A(a; 0, R)$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} c_n\, (z-a)^n \quad \Longrightarrow \quad \operatorname{Res}(f, a) = c_{-1}.$$

  That is, the residue of f at a is just the (-1)-th coefficient in the local Laurent expansion.

- **Residues and poles.** If a is a pole of order m, then

$$\operatorname{Res}(f, a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \big\{ (z-a)^m f(z) \big\}.$$

  In practice, this limit at the beginning never needs evaluating, since the product $(z - a)^m f(z)$ has an obvious interpretation at $z = a$.

- **The Residue Theorem.** Suppose that f is holomorphic on a simply connected domain U except at isolated singularities. If γ is a positively-oriented Jordan curve in U which does not pass through any singularities of f, then

$$\int_\gamma f(z)\, dz = 2\pi\, i \sum_{z \in \text{inside}(\gamma)} \operatorname{Res}(f, z).$$

  Note that while the sum appears infinite, the residue term is only nonzero at isolated singularities, of which there are only finitely many on the inside of γ.

- **Applications.** The Residue Theorem has two main applications:

  - *Evaluating complex line integrals.* If the function has easy Laurent expansions at its singularities, then line integrals can be computed with trivial ease without recourse to either the homotopy version of the CIT or the derivative version of the CIF.
  - *Evaluating indefinite real integrals.* If a real function is integrable over an improper interval, then the limit can be evaluated by *complexifying* the function, augmenting the interval to a closed curve, and then taking a limit.

The Argument Principle

- **Winding number.** The *winding number* of a closed curve γ about the point $a \notin \gamma$, intuitively, is the number of times that γ "wraps" around a in the positive direction. Technically, it is defined by

  $$\mathrm{Wind}(\gamma, a) := \frac{1}{2\pi\, i} \int_\gamma \frac{1}{z-a}\, dz.$$

  A useful identity involving composite curves is the following:

  $$\mathrm{Wind}(f \circ \gamma, a) = \frac{1}{2\pi\, i} \int_\gamma \frac{f'(z)}{f(z)-a}\, dz.$$

- **Total angular change.** Suppose γ is a closed curve and f is holomorphic and non-vanishing on a neighborhood of γ. The *total angular change* of f on γ, $\Delta_\gamma \arg f(z)$, intuitively measures the total change in the angle of the point $w = f(z)$ as z completes on circuit about γ.

  More precisely, it is defined as follows: let $z(t)$ be any smooth parametrization of γ defined of $[a, b]$, and let $\arg f\big(z(t)\big)$ denote any continuous argument function. Then

  $$\Delta_\gamma \arg f(z) := \arg f\big(z(b)\big) - \arg f\big(z(a)\big).$$

- **Angular change and winding numbers.** The total angular change of f over γ is the product of 2π times the winding number of $f \circ \gamma$ at 0, i.e.

  $$\Delta_\gamma \arg f(z) = 2\pi\, \mathrm{Wind}(f \circ \gamma, 0).$$

- **The Argument Principle.** Suppose f is analytic except for poles. Suppose that γ is a positively oriented Jordan curve which does not pass through any zero or pole of f. Then

  $$\frac{1}{2\pi\, i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{\substack{\text{zeros } a \\ \text{inside } \gamma}} \mathrm{order}(a) - \sum_{\substack{\text{poles } b \\ \text{inside } \gamma}} \mathrm{order}(b).$$

  Said differently, the winding number of f over γ is merely the difference between the total number of zeros (counting multiplicity) and the total number of poles (counting multiplicity) inside γ.