The Cauchy Integral Formula

Theorem. *Suppose U is a simply connected domain and $f : U \to \mathbb{C}$ is continuous and conservative. If γ is a positively-oriented Jordan curve around a point z^*, then*

$$f(z^*) = \frac{1}{2\pi\, i} \int_\gamma \frac{f(z)}{z - z^*}\, dz.$$

Proof. For $R > 0$, define the circular path

$$C_R := \left\{ |z - z^*| = R \right\} = \left\{ z^* + Re^{it} : 0 \le t \le 2\pi \right\}.$$

Observe that C_R is a positively-oriented Joradan curve. Also, let us define

$$K(z) := \frac{f(z)}{z - z^*},$$

which is $\mathbb{C}$-differentiable on $U \setminus \{z^*\}$. We wish to prove that

$$\int_\gamma K(z)\, dz = 2\pi\, i\, f(z^*).$$

Step 1. Pick a circle, any circle. Since z^* lies inside domain inside the curve γ, there exists a radius $R > 0$ such that

$$B(z^*, R) \subset \text{inside}(\gamma) \subset U.$$

In particular, this implies that for any radius $0 < r < R$, the circle C_r lies inside γ. Since the function $K(z)$ is $\mathbb{C}$-differentiable on the region between γ and C_r, which does not contain z^*, the homotopy property of conservative functions allows us to conclude that

$$\int_\gamma K(z)\, dz = \int_{C_r} K(z)\, dz \qquad \forall\, 0 < r < R.$$

In particular, the integral over any small circle is a constant, independent of the radius of the circle.

Step 2: Use continuity to give K the squeeze. Let $\epsilon > 0$. Since f is continuous at the point $z^* \in U$, there exists $0 < \delta < R$ such that

$$|z - z^*| < \delta \Longrightarrow \left| f(z) - f(z^*) \right| < \epsilon.$$

As before, we have

$$\int_{C_r} 1\, dz = 0,$$

while the parameterization $z(t) = z^* + r\, e^{it}$ for $0 \le t \le 2\pi$ yields

$$\int_{C_r} \frac{1}{z - z^*}\, dz = \int_0^{2\pi} \frac{1}{z(t) - z^*}\, z'(t)\, dt = \int_0^{2\pi} \frac{1}{r\, e^{it}}\, i\, r\, e^{it}\, dt = \int_0^{2\pi} i\, dt = 2\pi\, i.$$

Thus

$$
\begin{aligned}
\int_{C_r} \frac{f(z) - f(z^*)}{z - z^*}\, dz \\
&= \int_{C_r} \frac{f(z)}{z - z^*}\, dz - \int_{C_r} \frac{f(z^*)}{z - z^*}\, dz \\
&= \int_{C_r} K(z)\, dz - f(z^*) \int_{C_r} \frac{1}{z - z^*}\, dz \\
&= \int_{C_r} K(z)\, dz - 2\pi\, i\, f(z^*) \\
&= \int_{\gamma} K(z)\, dz - 2\pi\, i\, f(z^*).
\end{aligned}
$$

Now, if $0 < r < \delta$, then

$$
\begin{aligned}
\left| \int_{\gamma} K(z)\, dz - 2\pi\, i\, f(z^*) \right| = \left| \int_{C_r} \frac{f(z) - f(z^*)}{z - z*}\, dz \right| &\leq \int_{C_r} \left| \frac{f(z) - f(z^*)}{z - z*} \right| |dz| \\
&\leq \int_{C_r} \frac{\left| f(z) - f(z^*) \right|}{|z - z^*|}\, |dz| \leq \int_{C_r} \frac{\epsilon}{r}\, |dz| = 2\pi\, \epsilon.
\end{aligned}
$$

But $\epsilon > 0$ was abritary, whence letting $\epsilon \to 0$ above implies

$$
\left| \int_{\gamma} K(z)\, dz - 2\pi\, i\, f(z^*) \right| = 0,
$$

which completes the proof. □