The CR condition

Theorem. *Suppose $U \subset \mathbb{C}$ is open and $f = u + i\,v : U \to \mathbb{C}$ is a complex function. Then f is $\mathbb{C}$-differentiable at z if and only if f is $\mathbb{R}$-differentiable at z and satisfies the CR equations at z:*

$$\frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z), \qquad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z).$$

Proof. We will use the following notational conventions during the proof. We will denote a variable in italics, like $z = x + i\,y$, when we wish to emphasize the interpetation as a complex number; we will denote the same variable in boldface, like $\mathbf{z} = (x, y)$, when we wish to emphasize the interpretation as a real vector. Similarly, we will denote the norm by $|\cdot|$ for complex numbers and by $||\cdot||$ for real vectors.

($\Longrightarrow$) Suppose $f'(z)$ exists. Define the matrix

$$A := \begin{pmatrix} \operatorname{Re} f'(z) & -\operatorname{Im} f'(z) \\ \operatorname{Im} f'(z) & \operatorname{Re} f'(z) \end{pmatrix}.$$

We claim A is the derivative matrix $J_{\mathbf{f}}(\mathbf{z})$ for the vector mapping $\mathbf{f}$ at $\mathbf{z}$. To see this, observe first that for any vector $\mathbf{h} = (s, t)$,

$$\begin{aligned} A\,\mathbf{h} &= \big(s\ \operatorname{Re} f'(z) - t \operatorname{Im} f'(z), s\ \operatorname{Im} f'(z) + t\ \operatorname{Re} f'(z)\big) \\ &= \big(s\ \operatorname{Re} f'(z) - t \operatorname{Im} f'(z)\big) + i\big(s\ \operatorname{Im} f'(z) + t\ \operatorname{Re} f'(z)\big) = f'(z)\,h. \end{aligned}$$

Thus, for any $\mathbf{h} \neq \mathbf{0}$,

$$\begin{aligned} \frac{\big|\big|\mathbf{f}(\mathbf{z}+\mathbf{h}) - \mathbf{f}(\mathbf{z}) - A\,\mathbf{h}\big|\big|}{||\mathbf{h}||} &= \frac{\big|f(z+h) - f(z) - f'(z)h\big|}{|h|} \\ &= \left|\frac{f(z+h) - f(z) - f'(z)h}{h}\right| = \left|\frac{f(z+h) - f(z)}{h} - f'(z)\right|. \end{aligned}$$

Since this last expression tends to 0 as $h \to 0$, so must the first. Thus, $\mathbf{f}$ is $\mathbb{R}$-differentiable at $\mathbf{z}$. Moreover, since the Jacobian matrix of partial derivatives necessarily coincides with A, we must have

$$\frac{\partial u}{\partial x}(z) = \operatorname{Re} f'(z) = \frac{\partial v}{\partial y}(z), \qquad \frac{\partial u}{\partial y}(z) = -\operatorname{Im} f'(z) = -\frac{\partial v}{\partial x}(z),$$

so the CR equations hold.

($\Longleftarrow$) Suppose $\mathbf{f}$ is $\mathbb{R}$-differentiable at $\mathbf{z}$ and satisfies the CR equations. Hence, the Jacobian of $\mathbf{f}$ takes the form

$$J_{\mathbf{f}}(\mathrm{z}) = \begin{pmatrix} u_x(\mathrm{z}) & u_y(\mathrm{z}) \\ v_x(\mathrm{z}) & v_y(\mathrm{z}) \end{pmatrix} = \begin{pmatrix} u_x(\mathrm{z}) & -v_x(\mathrm{z}) \\ v_x(\mathrm{z}) & u_x(\mathrm{z}) \end{pmatrix}.$$

Set

$$\alpha := \frac{\partial u}{\partial x}(z) + i\,\frac{\partial v}{\partial x}(z).$$

We claim α is the derivative $f'(z)$ for the complex function f at z. To see this, observe first that for any complex number $h = s + i\,t$,

$$\begin{aligned}\alpha\, h &= \left(s\frac{\partial u}{\partial x}(z) - t\frac{\partial v}{\partial x}(z)\right) + i\left(s\frac{\partial v}{\partial x}(z) + t\frac{\partial u}{\partial x}(z)\right)\\ &= \left(s\frac{\partial u}{\partial x}(\mathrm{z}) - t\frac{\partial v}{\partial x}(\mathrm{z}), s\frac{\partial v}{\partial x}(\mathrm{z}) + t\frac{\partial u}{\partial x}(\mathrm{z})\right) = J_{\mathrm{f}}(\mathbf{z})\,\mathbf{h}.\end{aligned}$$

Thus, for any complex number $h \neq 0$,

$$\begin{aligned}\left|\frac{f(z+h)-f(z)}{h} - \alpha\right| &= \left|\frac{f(z+h)-f(z)-\alpha\, h}{h}\right|\\ &= \frac{\left|f(z+h)-f(z)-\alpha\, h\right|}{|h|} = \frac{\left|\left|\mathbf{f}(\mathbf{z}+\mathbf{h})-\mathbf{f}(\mathbf{z})-J_{\mathbf{f}}(\mathbf{z})\,\mathbf{h}\right|\right|}{||\mathbf{h}||}\end{aligned}$$

Since this last expression tends to 0 as $\mathbf{h} \to \mathbf{0}$, so must the first. Thus, f is $\mathbb{C}$-differentiable at z. $\square$