The Cauchy Estimates and Liouville's Theorem

Theorem. [Cauchy's Estimates] *Suppose f is holomrophic on a neighborhood of the closed ball $\overline{B(z^*, R)}$, and suppose that*

$$M_R := \max\left\{\left|f(z)\right| : |z - z^*| = R\right\}. \quad (< \infty)$$

Then

$$\left|f^{(n)}(z^*)\right| \le \frac{n!\, M_R}{R^n}.$$

Proof. According to the Cauchy Integral Formula, we have

$$f^{(n)}(z^*) = \frac{n!}{2\pi\, i} \int\limits_{|z-z^*|=R} \frac{f(z)}{(z - z^*)^{n+1}}\, dz.$$

Then

$$\begin{aligned}\left|f^{(n)}(z^*)\right| &= \left|\frac{n!}{2\pi\, i} \int\limits_{|z-z^*|=R} \frac{f(z)}{(z - z^*)^{n+1}}\, dz\right| \le \left|\frac{n!}{2\pi\, i}\right| \int\limits_{|z-z^*|=R} \left|\frac{f(z)}{(z - z^*)^{n+1}}\right| |dz| \\ &= \frac{n!}{2\pi} \int\limits_{|z-z^*|=R} \frac{|f(z)|}{|z - z^*|^{n+1}}\, |dz| \le \frac{n!}{2\pi} \int\limits_{|z-z^*|=R} \frac{M_R}{R^{n+1}}\, |dz| = \frac{n!}{2\pi}\, \frac{M_R}{R^{n+1}}\, 2\pi R = \frac{n!\, M_R}{R^n}. \square\end{aligned}$$

Corollary. [Liouville's Theorem] *A bounded entire (i.e. everywhere differentiable) function is constant.*

Proof. Suppose $f : \mathbb{C} \to \mathbb{C}$ is everywhere differentiable and is bounded above by M, i.e. $|f(z)| \le M$ for every $z \in \mathbb{C}$.

Fix an arbitary z^*. Since f is holomorphic everywhere, it is in particular holomorphic on a neighborhood of $\overline{B(z^*, R)}$ for any value of $R > 0$. By the Cauchy Estimates, since

$$M_R := \max\left\{\left|f(z)\right| : |z - z^*| = R\right\} \le M$$

for any $\mathbb{R}$, we have

$$\left|f'(z^*)\right| \le \frac{M_R}{R} \le \frac{M}{R} \quad \forall\, R > 0.$$

Since the expression on the left is a nonnegative constant, letting $R \to \infty$ on the right yields

$$0 \le \left|f'(z^*)\right| \le 0,$$

whence $f'(z^*) = 0$.

But z^* was arbitrary, so $f'(z) \equiv 0$ on $\mathbb{C}$. But then $f(z)$ is necessarily a constant (as shown in the homework). $\square$