Complex antiderivatives and Goursat's Theorem

Theorem. *Suppose U is open and $f : U \to \mathbb{C}$ is continuous. Then f possesses a complex antiderivative on U if and only if*

$$\int_\Delta f\,dz = 0$$

for any triangular path Δ in U.

Before getting to the proof, there is an immediate corollary:

Corollary. *Suppose U is a simply-connected domain and $f : U \to \mathbb{C}$ is complex differentiable. Then f possesses a complex antiderivative on U.*

Proof of corollary. If f is a $\mathbb{C}$-differentiable function on a simply connected domain, then Cauchy's Theorem implies $\int_\gamma f\,dz = 0$ for any closed curve γ, and in particular for triangular γ. □

Proof of theorem. ($\Longrightarrow$) Suppose $F'(z) = f(z)$, and let γ be any smooth curve from in U a point α to a point β. If we parametrize γ by $z = z(t)$ with $a \le t \le b$, then by the *real* Fundamental Theorem of Calculus (FTC), appliyed to the real and imaginary parts of the integral, we have

$$\begin{aligned}\int_\gamma f\,dz &= \int_a^b f\big(z(t)\big)z'(t)\,dt = \int_a^b F'\big(z(t)\big)z'(t)\,dt \\ &= \int_a^b \frac{d}{dt}\big\{F\big(z(t)\big)\big\}\,dt = F\big(z(b)\big) - F\big(z(a)\big) = F(\beta) - F(\alpha).\end{aligned}$$

This is a complex analogue of the first part of the real FTC.

In particular, if $\Delta = [z_1, z_2, z_3, z_1]$, then

$$\begin{aligned}\int_\Delta f\,dz &= \int_{[z_1,z_2]} f\,dz + \int_{[z_2,z_3]} f\,dz + \int_{[z_3,z_1]} f\,dz \\ &= \big(F(z_2) - F(z_1)\big) + \big(F(z_3) - F(z_2)\big) + \big(F(z_1) - F(z_3)\big) = 0.\end{aligned}$$

($\Longleftarrow$) Assume U is simply connected, and fix a $z_0 \in U$. Define $F : U \to \mathbb{C}$ by

$$F(z) := \int_{P[z_0,z]} f(\zeta)\,d\zeta,$$

where $P[z_0, z]$ is any polygonal path connecting z_0 to z.

Observe that F is well-defined. Since U is open and connected, it is polygonally arcwise connected, so a polygonal path γ from z_0 to z exists. Moreover, if η is another such path, then $\gamma - \eta$ is a closed polygonal path. Since such a path can be decomposed as a sum of triangular paths, we conclude

$$0 = \int_{\gamma-\eta} f(\zeta)\,d\zeta = \int_\gamma f(\zeta)\,d\zeta - \int_\eta f(\zeta)\,d\zeta.$$

Hence, the two lines integrals over γ and η agree.

It remains to show that F is an antiderivative of f, i.e. that

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z) \qquad \forall\, z \in U.$$

Fix z, and let $\epsilon > 0$. Since U is open, there exits an $r > 0$ such that $B(z,r) \subset U$. Moreover, since f is continuous at z, there exists $0 < \delta < r$ such that

$$|w - z| < \delta \Longrightarrow \big|f(w) - f(z)\big| < \epsilon.$$

Now, fix a polygonal path γ from ζ to z. If $0 < |h| < r$, then $z+h \in B(z,r)$, and so $\gamma + [z, z+h]$ is a polygonal path from ζ to $z+h$. Hence,

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\left(\int\limits_{\gamma+[z,z+h]} f(\zeta)\, d\zeta - \int_\gamma f(\zeta)\, d\zeta \right) = \frac{1}{h} \int\limits_{[z,z+h]} f(\zeta)\, d\zeta.$$

On the other hand, using the first half of this theorem, since 1 has the antiderivative ζ,

$$\int\limits_{[z,z+h]} 1\, d\zeta = \zeta \Big|_z^{z+h} = (z+h) - z = h,$$

whence

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h} \int\limits_{[z,z+h]} f(\zeta)\, d\zeta - f(z) \left[\frac{1}{h} \int\limits_{[z,z+h]} 1\, d\zeta \right] = \frac{1}{h} \int\limits_{[z,z+h]} f(\zeta) - f(z)\, d\zeta.$$

Thus, if $|h| < \delta$, then

$$\begin{aligned} \left| \frac{F(z+h)-F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int\limits_{[z,z+h]} f(\zeta) - f(z)\, d\zeta \right| \le \frac{1}{|h|} \int\limits_{[z,z+h]} \big|f(\zeta) - f(z)\big|\, |dz| \\ &< \frac{1}{|h|} \int\limits_{[z,z+h]} \epsilon\, |dz| = \frac{1}{|h|} \cdot \epsilon\, |h| = \epsilon, \end{aligned}$$

which completes the proof. □