The Cauchy-Goursat Theorem

Theorem. *Suppose U is a simply connected domain and $f : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable. Then*

$$\int_\Delta f\,dz = 0$$

for any triangular path Δ in U.

Proof. Let Δ be a triangular path in U, i.e. a closed polygonal path $[z_1, z_2, z_3, z_1]$ with three points $z_1, z_2, z_3 \in U$. Let

$$M = \left|\int_\Delta f\,dz\right|, \qquad \ell = \text{perimeter}(\Delta).$$

We show $M = 0$.

Step 1: Divide and conquer. By connecting the three midpoints of each segment, we can divide Δ into four smaller, similar triangles, $\Delta_a, \Delta_b, \Delta_c, \Delta_d$. If we orient each subtriangle the same way as Δ, then after cancelling the three segments crossed twice we get

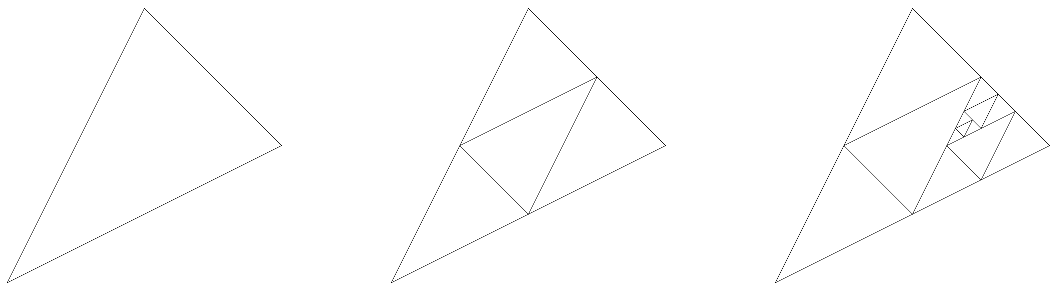
$$\int_\Delta f\,dz = \int_{\Delta_a} f\,dz + \int_{\Delta_b} f\,dz + \int_{\Delta_c} f\,dz + \int_{\Delta_d} f\,dz.$$

It must therefore follow that for one of the triangles, call it Δ_1, we must have

$$\left|\int_{\Delta_1} f\,dz\right| \geq \frac{M}{4},$$

for otherwise

$$M = \left|\int_\Delta f\,dz\right| \leq \left|\int_{\Delta_a} f\,dz\right| + \left|\int_{\Delta_b} f\,dz\right| + \left|\int_{\Delta_c} f\,dz\right| + \left|\int_{\Delta_d} f\,dz\right| < M.$$



Step 2: Get the limit point z^*. Repeat this argument on Δ_1 now. We obtain, by induction, a sequence of triangles (Δ_n) with the following properties:

$$\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots, \qquad \text{perimeter}(\Delta_n) = \frac{\ell}{2^n}, \qquad \left|\int_{\Delta_n} f\,dz\right| \geq \frac{M}{4^n}.$$

Since the triangles bounded by Δ_n are compact and their diameters (which are bounded above by their perimeters) tend to 0, we conclude there exists a unique point z^* contained inside every Δ_n.

Step 3: Use differentiability to give M the squeeze. Now, let $\epsilon > 0$ be arbitrary. Since f is analytic at the point $z^* \in U$, there exists $\delta > 0$ such that

$$|z - z^*| < \delta \Longrightarrow \left| \frac{f(z) - f(z^*)}{z - z*} - f'(z*) \right| < \epsilon \Longrightarrow \big| f(z) - f(z^*) - f'(z*)(z - z^*) \big| < \epsilon |z - z^*|.$$

Choose n so large that perimeter$(\Delta_n) < \delta$.

Since 1 and z have complex antiderivatives defined for all of $\mathbb{C}$, the complex Fundamental Theorem of Calculus implies

$$\int_{\Delta_n} 1\, dz = \int_{\Delta_n} z\, dz = 0,$$

whence

$$\begin{aligned}
&\int_{\Delta_n} f(z) - f(z^*) - f'(z*)(z - z^*)\, dz \\
&\quad = \int_{\Delta_n} f(z)\, dz - \int_{\Delta_n} f(z^*)\, dz - \int_{\Delta_n} f'(z^*)(z - z^*)\, dz \\
&\quad = \int_{\Delta_n} f(z)\, dz - \big(f(z^*) + f'(z^*) z^* \big) \int_{\Delta_n} 1\, dz - f'(z^*) \int_{\Delta_n} z\, dz \\
&\quad = \int_{\Delta_n} f(z)\, dz - 0 - 0 = \int_{\Delta_n} f(z)\, dz.
\end{aligned}$$

Now, observe that for any $z \in \Delta_n$,

$$|z - z^*| < \text{perimeter}(\Delta_n) = \frac{\ell}{2^n} < \delta,$$

so that

$$\begin{aligned}
\frac{M}{4^n} &\le \left| \int_{\Delta_n} f(z)\, dz \right| = \left| \int_{\Delta_n} f(z) - f(z^*) - f'(z^*)(z - z^*)\, dz \right| \\
&\le \int_{\Delta_n} \big| f(z) - f(z^*) - f'(z^*)(z - z^*) \big|\, |dz| \le \int_{\Delta_n} \epsilon |z - z^*|\, |dz| \\
&\le \int_{\Delta_n} \epsilon \cdot \frac{\ell}{2^n}\, |dz| = \epsilon \cdot \frac{\ell}{2^n} \cdot \frac{\ell}{2^n} = \frac{\epsilon\, \ell^2}{4^n}.
\end{aligned}$$

Therefore, examining either end of the intequality, we conclude

$$0 \le M \le \epsilon\, \ell^2.$$

But $\epsilon > 0$ was abritrary, whence letting $\epsilon \to 0$ above implies $M = 0$. $\square$