Introduction to Geometric Langlands I (D. Nadler) 1

Freiburg, 30 Jul 2012

G – reductive over $\mathbb{C}$

Aside: (from topology)

$G \rightsquigarrow BG$ classifying space

$\Omega_x X \;\leftsquigarrow\; X \ni x$ connected space

↑

based loop space

Notation $G \supset B \supset T$; $B \supset N$; $W = N(T)/T$ (weyl group)

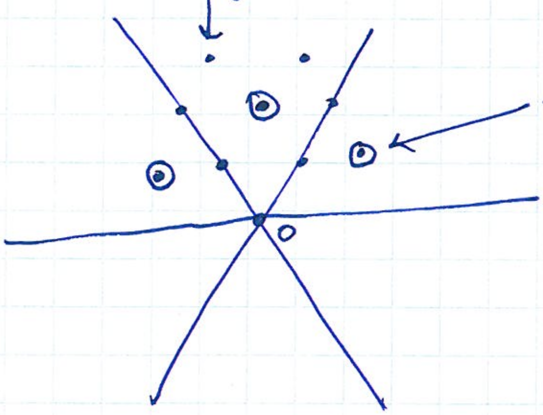
$\downarrow \quad \wr\wr$

$H = B/N$ (universal cartan)

$G = SL_3$

weights

$\mathfrak{t}^* \supset \Lambda_T^\vee$ ← weight lattice



+ve roots

$R^{\vee,+}$

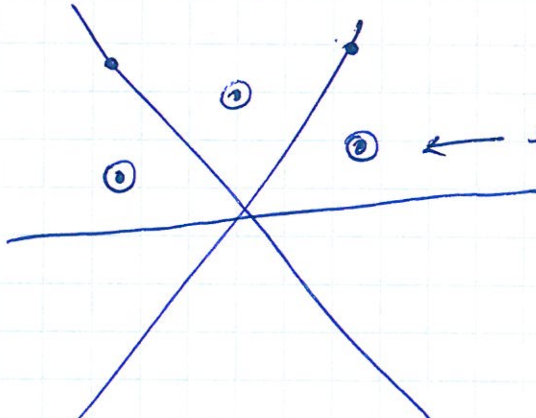
$\Lambda_T^{\vee,dom}$ – dominant wts.

$G = PGL_3 = SL_3/\mathbb{Z}$

coweight lattice

↓

$\Lambda_T \subset \mathfrak{t}$



+ve roots

R^+

Towards Hecke algebras for G, LG loop group

Geometry of 'coset multiplication $B\backslash G/$

$$B\backslash G/B \times B\backslash G/B \longrightarrow B\backslash G/B'$$

What is $B\backslash G/B$?

G/B flag variety ; $B\backslash G/B \approx$ B-equivariant geometry of G/B

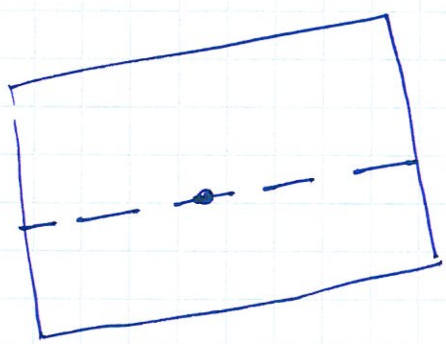
Exercise $B\backslash G/B \simeq \underbrace{BB \times_{BG} BB}_{\text{classifying spaces}}$

All questions make sense in the context of

$X \times_Y X$, $X \longrightarrow Y$ proper.

Pictures of $B\backslash G/B$

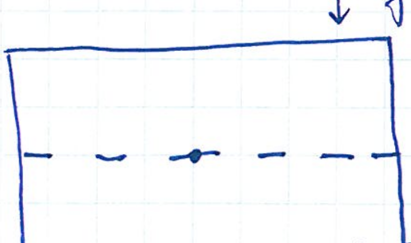
$G = SL_3$



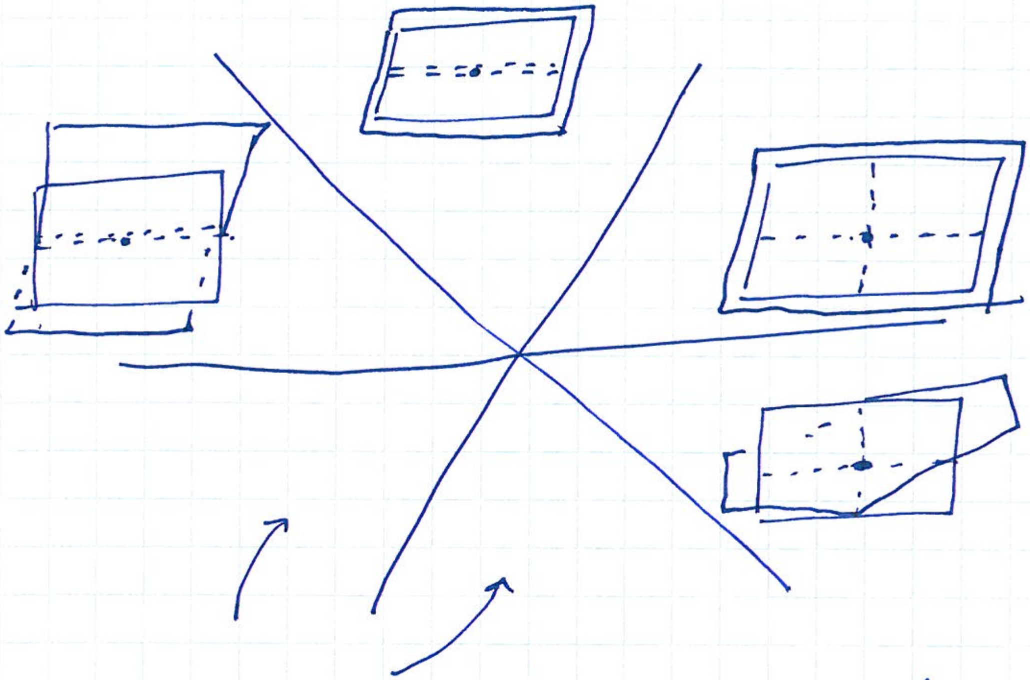
$\leftarrow$ G/B

$B\backslash G/B$

standard reference flag



given another flag, only care about relative position w/ respect to standard reference flag



and my drawing skills are awful (fill in w/ other remaining relative positions)

Two questions

1) How are the W points of $B\backslash G/B$ glued together?

2) What does group multiplication look like in terms of $B\backslash G/B$?

pts of $B\backslash G/B$ $\leadsto$ B-orbits in G/B

Exercise Show all Schubert varieties are smooth for SL_3. What are they?

Loop Groups

$$LG = G(K)$$

$D^\times$ = "circle" = spec K , $K = \mathbb{C}((t))$

Two natural parabolics — max $L^+G = G(\mathcal{O})$; $\mathcal{O} = \mathbb{C}[[t]]$

— min I - Iwahori ; $\begin{array}{ccc} L^+G & \xrightarrow{ev_0} & G \\ \cup & \square & \cup \\ I & \longrightarrow & B \end{array}$

$D = \mathrm{spec}(\mathbb{C}[[t]])$

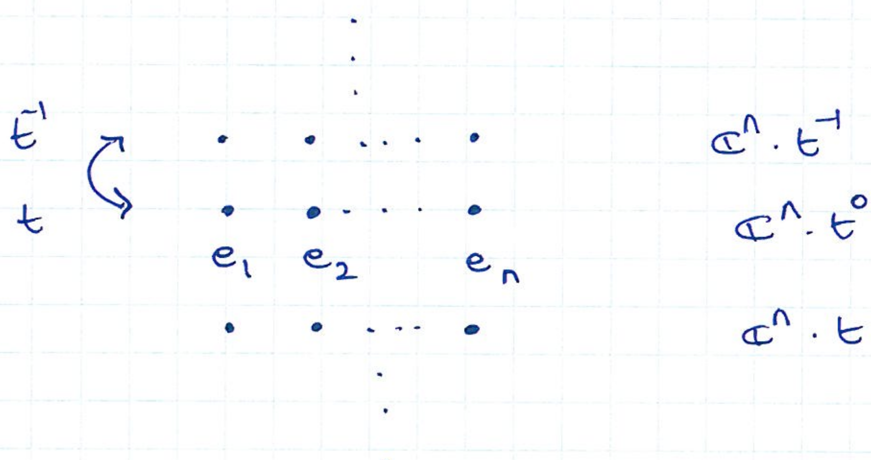
Def (Affine) flag variety for G : $\mathcal{Fl} = LG/I$

affine Grassmannian " : $Gr_G = LG/L^+G$

$\mathcal{Fl} \longrightarrow Gr_G$ (fibration w/ fibre G/B)

$G = GL_n$

$$LG \curvearrowright K^n$$



what does $GL_n(\mathcal{O})$ fix?

It fixes $\mathcal{O}^n$.

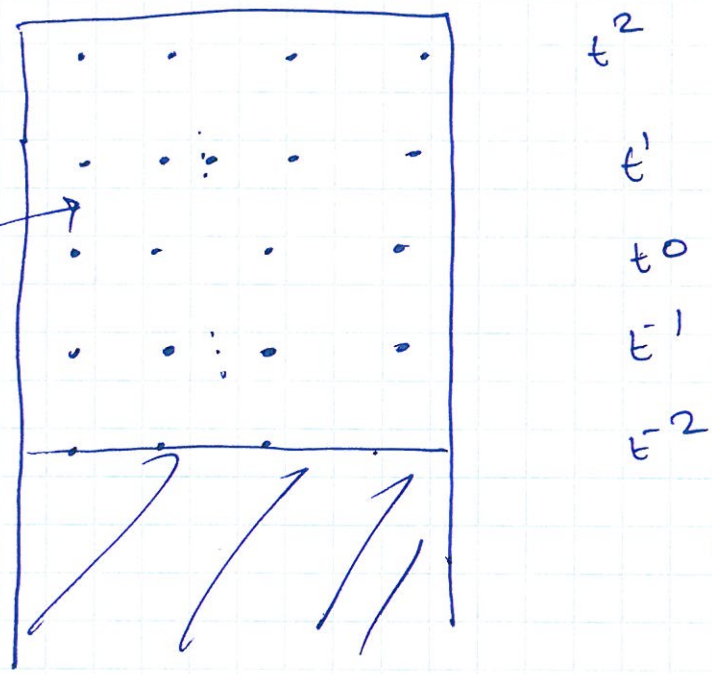
Actually $GL_n(\mathcal{O}) = \mathrm{Fix}(\mathcal{O}^n)$

Exercise / Prop $Gr_{GL_n} = \left\{ \begin{array}{l} W \subset K^n \\ \text{subspace} \end{array} \middle| \begin{array}{l} 1)\ t\cdot W \subset W \\ 2)\ \exists N \gg 0, \text{ s.t. } t^N \mathcal{O}^n \subset W \subset t^{-N}\mathcal{O}^n \end{array} \right\}$

Prop $Gr_{GL_n} = \bigcup_{k=0}^{\infty} Gr^k_{GL_n}$, $Gr^k_{GL_n} \subset Gr_{GL_n}$ (take $N=k$ above) 5

Prop $Gr^k_{GL_n}$ is a projective variety.

$Gr^2_{GL_n}$



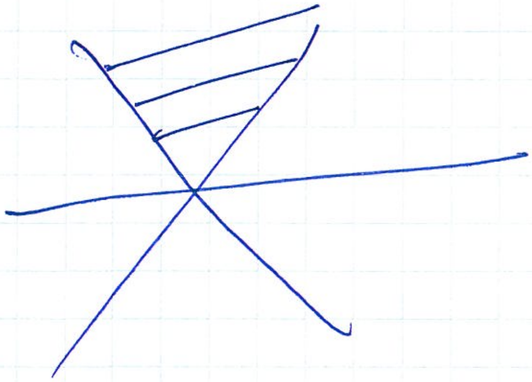
subspaces in $t^{-2}\mathcal{O}^n / t^2\mathcal{O}^n$

__Upshot__ : springer fibres!

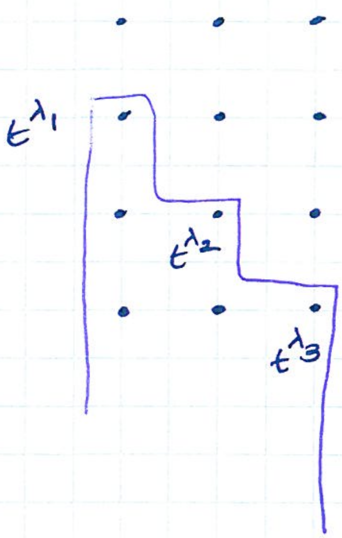
__Double cosets__ $L^+G \backslash LG / L^+G$ is L^+G equivariant geometry of Gr_G

__Cartan decomposition__

$$L^+G \backslash LG / L^+G \simeq \Lambda_G^{dom} \quad \text{dominant coweights}$$



$G = GL_3$



$\lambda_1 \geq \lambda_2 \geq \lambda_3$

<u>Exercise</u> $\lambda \in \Lambda_G^{dom} \rightsquigarrow Gr_G^{\lambda} \subset Gr_G$

$\uparrow$ L^+G orbit through t^{λ}

Show Gr_G^{λ} is a vector bundle over a partial <u>flag variety.</u>

What is the partial flag variety?
dim? of the vector bundle?

$\uparrow$
= distance of wt. from 0.

<u>Exercise</u> What is $\pi_0 Gr_G$?

<u>Exercise</u> $Gr_T \simeq \mathbb{Z}^{\dim T}$

<u>Exercise</u> Consider Gr_{GL_2}. Show $\overline{Gr}^{\lambda}_{GL_2}$ is rationally smooth.

<u>One last point of view</u> Let $K \subset G$ max compact subgroup

$LG \simeq L^+G \times \Omega^{poly}_{based}(K)$. i.e., $Gr_G = \Omega^{poly}_{based}(K)$

<u>Exercise</u> $G = PGL_2 = SO_3$, so $K = SO_3(\mathbb{R})$

Ω^{poly}_{based} = free group on S^2 / pr. $\alpha(pt) = 1$

$\uparrow$
antipode

Interpret lecture in terms of this!