Introduction to geometric Langlands II D. Nadler ①

(Freiburg, 31 Jul 2012)

categorical linearization of topology

Geometry

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

linearization allows us to see 'quantum symmetries'

sheaves allow us to linearize and see the full structure of spaces w/ symmetry.

Exercise G reductive group/$\mathbb{C}$

$QCoh(BG) \simeq Rep(G)$ show you can recover G from $Rep(G)$.

constructible sheaves

$Sh(pt) = Sh(pt, \mathbb{C})$ = differential graded bounded derived category.

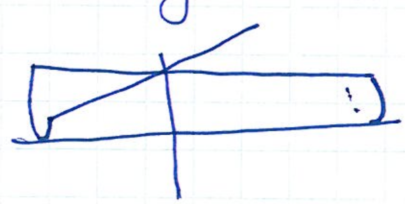
objects - usual complexes

morphisms - full Hom complex localized at quasi-isomorphisms

X - algebraic variety, S stratification

0) strata are smooth

1) $Homeo(X, S)$ acts transitively on each stratum

a non-example (Whitney's umbrella) 

2) Normal slice to any pt in a stratum is cone over (X', S') of smaller dimension ②

Examples 1) Whitney stratifications

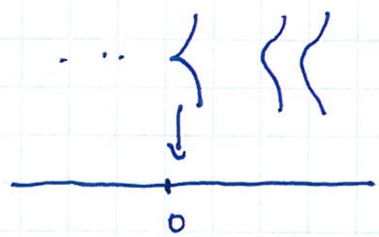
2) varieties w/ group actions, finitely many orbits

$Sh_S(X)$ $\quad D_S$ bounded constructible derived category

Nearby and vanishing cycles

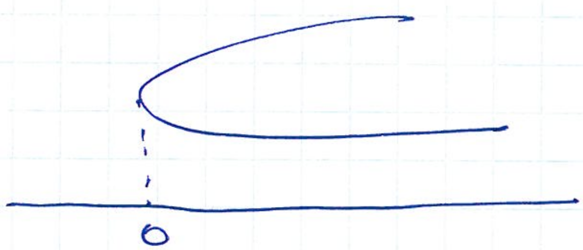
X - complex algebraic variety / $\mathbb{C}$

$f : X \longrightarrow \mathbb{A}^1$



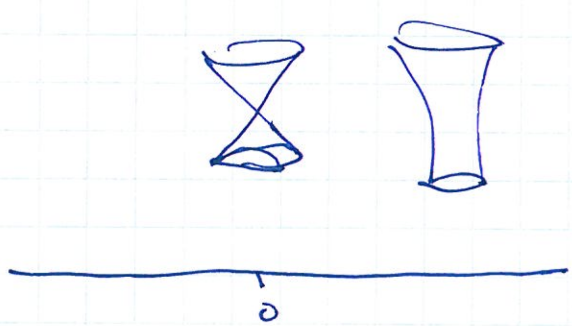
Example

0)



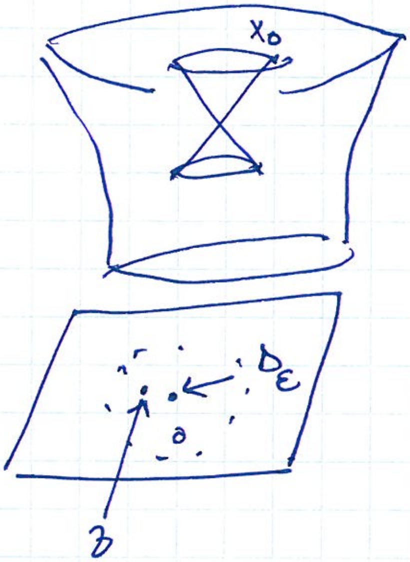
$$\mathbb{C} \xrightarrow{f(x) = x^2} \mathbb{C}$$

1)



$$\mathbb{C}^2 \xrightarrow{f(x,y) = x^2 + y^2} \mathbb{C}$$

<u>Collapse map</u> There exists an (appropriately unique) retraction $r : X_{D_\varepsilon} \longrightarrow X_0$



<u>Definition</u> ~~given $F \in Sh(X \setminus X_0)$~~

suppose $F \in Sh(X)$ satisfies

$f_* F \in Sh_S(\mathbb{C})$, $S = \{0, \mathbb{C}^*\}$

The <u>nearby cycles</u> of F is

$$\Psi F = r_* F|_{X_z}$$

as you move z around in a loop we get the monodromy transformation $\pi_1(S^1) \circlearrowright \Psi F$

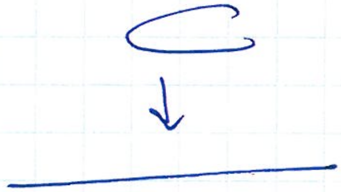
<u>Definition</u> vanishing cycles of F

$$F|_{X_0} \longrightarrow \Psi F \longrightarrow \Phi F \rightsquigarrow$$

vanishing cycles is also equipped w/ a compatible monodromy transformation

<u>Example</u>

0) Take F to be the constant sheaf $\underline{\mathbb{C}}$

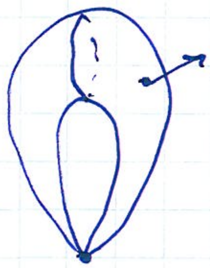


$$\begin{tikzcd} F|_{X_0} \arrow[r] \arrow[d, equal] & \Psi F \arrow[r] & \Phi F \arrow[r, squiggly] & {} \\ \underline{\mathbb{C}}_{\{0\}} \arrow[r] & \mathbb{C} \oplus \mathbb{C} \arrow[r] \arrow[loop below, "\sigma"] & \underline{\mathbb{C}}_{\{0\}} \arrow[r, squiggly] \arrow[loop below, "-1"] & {} \end{tikzcd}$$

1) F = constant sheaf again

$$F|_{X_0} \longrightarrow \Psi F \longrightarrow \Phi F \rightsquigarrow$$

Perverse sheaves $\quad P_S(X) \subseteq Sh_S(X)$



← generic covector at pt.

$x \in X$, $f : X \to \mathbb{C}$

$f(x) = 0 \qquad df|_x$ generic

$\Phi F|_x$ $\quad$ $\mathrm{Perv}_S(X)$ full subcategory s.t.

$\ \ \|$ $\qquad\qquad$ $M_{x,\xi}(F)$ are concentrated in deg. 0

$M_{x,\xi}(F)$ local Morse group, $\xi = df|_x$

Exercise $\quad$ Describe sheaves on $\mathbb{C}$ with $S = \{0, \mathbb{C}^*\}$