Introduction to geometric Langlands IV (D. Nadler) ①

(Freiburg, 2 Aug 2012)

Reminders

$\mathcal{H}_G = sh(B\backslash G/B)$; $\mathcal{H}_{LG} = sh(I\backslash LG/I)$;

$\mathcal{H}_{LG}^{sph} = sh(L^+G\backslash LG/L^+G)$.

sitting inside each of these is the abelian category of perverse sheaves, and inside this category are IC complexes.

Indexing sets for IC objects:

$\mathcal{H}_G \rightsquigarrow W$ finite Weyl group (IC_w)

$\mathcal{H}_{LG} \rightsquigarrow W_{aff}$ affine Weyl group (IC_w)

$\mathcal{H}_{LG}^{sph} \rightsquigarrow \Lambda_G^{dom} = \Lambda_G/W$ dominant coweights (IC_λ)

characterization of $IC_{Y,\mathcal{L}}$ ($Y \subseteq X$, $\mathcal{L}$ local system on Y)

strata

for all $S' \subseteq \bar{S}$, $S' \neq S$

$i_x^* IC_{S,\mathcal{L}}$ concd. in degrees $< -\dim S'$

Note:

$IC_{S,\mathcal{L}}|_S = \mathcal{L}[\dim S]$

$x \in S'$

and dually

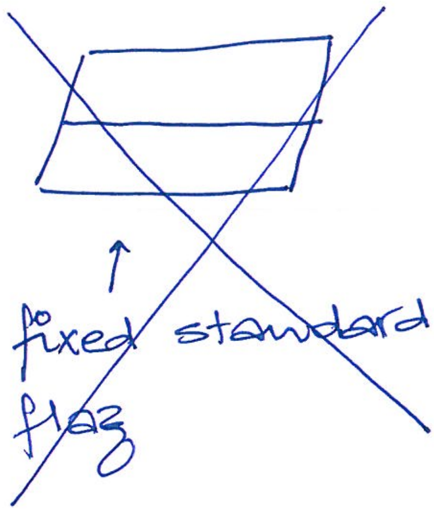
$i_x^! IC_{S,\mathcal{L}}$ concd. in degrees $> \dim S'$

convolution

$$(X\times_Y X)\times(X\times_Y X) \xleftarrow{p_{12}\times p_{23}} X\times_Y X\times_Y X \xrightarrow{p_{13}} X\times_Y X$$

$\rightsquigarrow$ monoidal structure on the Hecke categories

~~warmup: 'cartoon'~~



warmup $\mathcal{H}_G$ say for $G = SL_2$

IC_e = unit

$IC_s * IC_s = IC_s[1] \oplus IC_s[-1]$

$\mathcal{H}_{SL_3}$

$IC_s * IC_t = IC_{st}$

$IC_t * IC_s = IC_{ts}$

Thm (Geometric Satake)

miracle 1: convolution in $\mathcal{H}^{sph}_{LG}$ preserves perverse sheaves
(not hard to prove, follows from an analysis of dimensions)

miracle 2: $P^{sph}_{LG} \subset \mathcal{H}^{sph}_{LG}$ is commutative

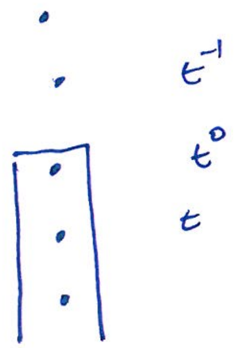
$$\rightsquigarrow \quad P^{sph}_{LG} \simeq Rep_{fd}(G^\vee)$$

example $G = GL_1$

$Gr_{GL_1} \cong \mathbb{Z} \simeq \Lambda_{GL_1}$

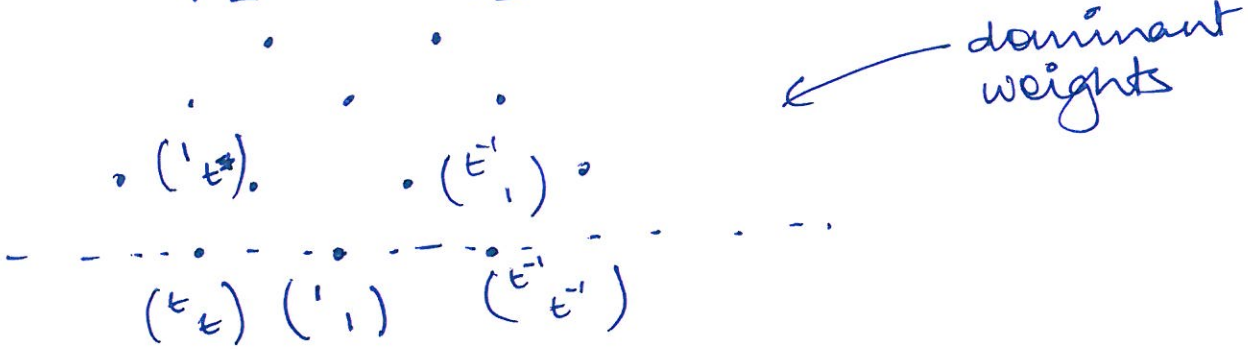
$P^{sph}_{LGL_1} \simeq Vect_{\mathbb{Z}} \simeq Rep(GL_1)$

convolution $\rightsquigarrow$ group addition



example $G = GL_2$

$\Lambda^{dom}_{GL_2} \subset \Lambda_{GL_2} \simeq Hom(\mathbb{C}^*, T)$



what completes the following square:

$$\begin{tikzcd} P^{sph}_{LG} \arrow[r, "\sim", "\text{satake}"'] \arrow[d, "?"'] & \mathrm{Rep}_{fd}(G^\vee) \arrow[d, "\mathrm{res}"] \\ P^{sph}_{LT} \arrow[r, "\sim"] & \mathrm{Rep}_{fd}(T^\vee) \end{tikzcd}$$

these are tensor functors.

$T \subset G$ maximal torus

$B \subset G$ Borel

$\cup$

N

$N(K) \hookrightarrow Gr_G$ $\frac{\infty}{2}$ - orbits (infinit dim and codim)

Iwasawa : $N(K) \backslash LG / L^+G \simeq \Lambda_G$

notation : write $N_\lambda \subset Gr_G$ for the orbit through λ, i.e., $N_\lambda = N(K) \cdot \lambda$

$$Gr_G \xleftarrow{p} \coprod_\lambda N_\lambda \xrightarrow{q} \{\lambda\} \in Gr_T$$

Prop res^G_T corresponds to $q_! p^*$

(satake transform)

why are all these things perverse?

$\leadsto$ vanishing cycles!