Introduction to Geometric Langlands ~~IV~~ V (D. Nadler) ①

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Reminders

$P^{sph}_{{}^LG}$ - spherical perverse Hecke category

$\cap$ $\mathcal{H}^{sph}_{{}^LG}$

$\mathcal{H}_{{}^LG}$

also have $\mathcal{H}_{{}^LG}$

Two miracles: convolution preserves perverse sheaves in $\mathcal{H}^{sph}_{{}^LG}$

$P^{sph}_{{}^LG}$ is commutative

Bundle interpretations

warmup: $B\backslash G/B \simeq BB \times_{BG} BB$

BG = G-bundles on a pt.

to trivialize a G bundle just need to pick a point (then automorphisms are just G).

$BG = pt/G$ ~~(G-bundles on a pt + 2 reductions~~

$G = pt \times_{BG} pt$ = G-bundles on pt + 2 trivializations

$BB \times_{BG} BB$ = 2 B-bundles on pt + identification of induced G-bundles

$G/B = pt \times_{BG} BB$ = G-bundles + trivialization
+ reduction to B

Now to LG

LG = G-bundles on $D^\times$ w/ 2 trivializations

$Gr_G = LG/L^+G$ ~~≃ G-bundles on $D^\times$ + trivializatio-n + trivialization that extends~~ over D

= G-bundles on $D^\times$
+ trivialization over $D^\times$
+ bundle extension over D

~~TAC~~ $L^+G \backslash LG / L^+G$ = 2 G-bundles on D
+ identification over $D^\times$

Ex $Gr_G \simeq$ G-bundles on a curve C
+ trivialization away from some fixed pt $x \in C$.

Ex G- semisimple

$Bun_G(C) \simeq \Gamma_c \backslash LG / L^+G \leftarrow L^+G \backslash LG / L^+G$

$c \in C$, $\Gamma_c = maps(C - \{c\}, G)$

Use the bundle interpretation to reformulate convolution.

Beilinson-Drinfeld Grassmanian

C, $n \geq 0$
$\uparrow$
fixed curve

$$\mathrm{Gr}^{(n)}_G = \begin{array}{l} G\text{-bundles on } C \\ + n \text{ pts. of } C,\ c_1, \ldots, c_n, \\ + \text{trivializations on} \\ \quad C - \{c_1, \ldots, c_n\} \end{array}$$

$$\begin{array}{ccc} \mathrm{Gr}^{(n)}_G & \longrightarrow & C^n \\ \cup & & \cup \\ \mathrm{Gr}^{(n)}_G\big|_{\underline{c}} & & \underline{c} \end{array}$$

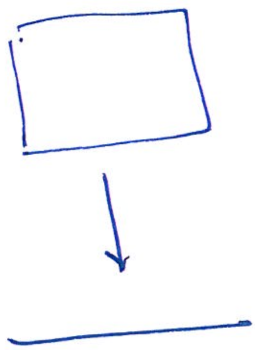
$\wr\wr$

$\mathrm{Gr}^{|\underline{c}|}_G = |\underline{c}|$ copies of Gr_G ; $|\underline{c}| = \#$ of distinct pts in $\underline{c} = (c_1, \ldots, c_n)$

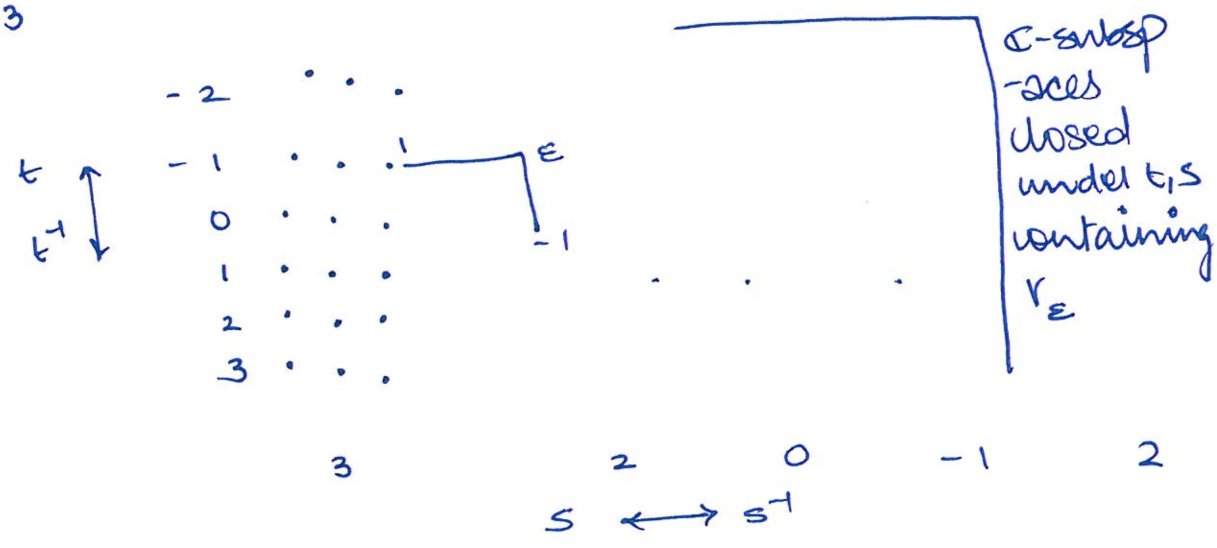
Grassmanian style description of $\mathrm{Gr}^{(n)}_G$

For simplicity, take $C = \mathbb{A}^1$ and $n = 2$

$$\mathbb{A}^2 \longrightarrow \mathbb{A}^1$$
$$(t, s) \longmapsto t - s$$



$G = GL_3$



Ex $\varepsilon \neq 0 \rightsquigarrow Gr \times Gr$

$\varepsilon = 0 \rightsquigarrow Gr$

Thm Convolution in P^{sph}_{LG} is equivalent to nearby cycles along collapse in BD-Grassmannian

$IC_\lambda * IC_\mu \simeq \bigoplus_\nu IC_\nu$

$\psi(IC_\lambda \boxtimes IC_\mu \boxtimes \mathbb{C}_{/\mathbb{A}-\{0\}})$

Remark 1) ψ preserves perverse sheaves in general

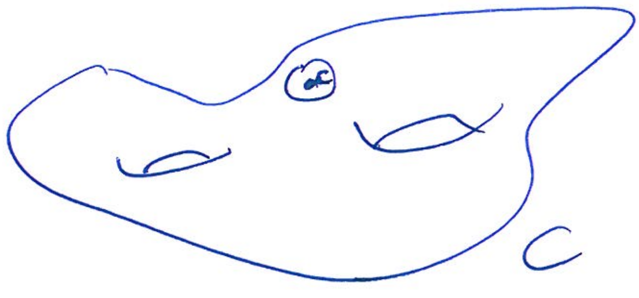
2) pts. on C can move around each other $\rightsquigarrow$ commutativity

finally Intro to the affine Hecke category

$\mathcal{H}_{LG} = Sh(I \backslash LG / I)$

objects of $\mathcal{H}_{LG}$ are kernels for modifications of

G-bundles on C + reduction to B at pt. $c \in C$



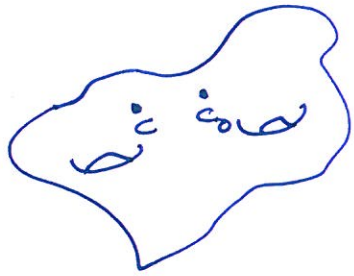
Structure 1.

$$Z : P^{sph}_{LG} \longrightarrow \mathcal{H}_{LG}$$

'central homomorphism'

consider the moduli space $\widetilde{Fl}_G$

= G-bundles over C, + pt $c \in C$,
+ trivialization away from c
+ flag (reduction to B) at c_0



$$\begin{tikzcd} \widetilde{Fl}_G \arrow[r] & C \\ Gr_G \times G/B \arrow[u, hook] \arrow[r] & c \neq c_0 \arrow[u, hook] \end{tikzcd}$$

$$Fl_G \longrightarrow c_0$$

now take nearby cycles!

$$Z(IC_\lambda) = \Psi(IC_\lambda \boxtimes \mathbb{C}_{B/B} \boxtimes \mathbb{C}_{C-\{c_0\}})$$

First picture $\quad S_\lambda = \mathbb{P}^1$

structure 2

spectral interpretation of $\mathcal{H}_{LG}$

$$
\begin{tikzcd}
\mathcal{H}_{LG} \arrow[rr, "\sim", "\text{monoidal}"'] & & \mathrm{Coh}\left({}^{St}G^{\vee}/G^{\vee}\right) \\
P^{sph}_{LG} \arrow[u] \arrow[rr, "\text{satake}", "\sim"'] & & \mathrm{Coh}(BG^{\vee}) \arrow[u]
\end{tikzcd}
$$

monodromy $\mapsto Z$

$\leftarrow\!\!\shortmid$ universal endomorphism of $W^{\vee}$