

Introduction to Geometric Langlands I (T. Pantev) ①

(Freiburg, 30 Jul 2012)

Formulation of the Geometric Langlands Conjecture (this is the goal)

Background data:

C - smooth projective curve/ \mathbb{C} , genus $g > 1$

G - complex reductive group

$T \subset B \subset G$; $\text{char}(T) = \text{Hom}(T, \mathbb{C}^*)$, $\text{wchar}(T) = \text{Hom}(\mathbb{C}^*, T)$

$\text{root}_G = \text{char}(T) = \text{weight}_G$
 $\text{wweight}_G = \text{wchar}(T) = \text{wroot}_G$

Two reductive groups, $G, {}^L G$ are Langlands dual if

$$\text{char}(G) \cong \text{wchar}({}^L G)$$

moduli objects: Bun = moduli stack of principle algebraic G -bundles on C

Loc = moduli stack of principle algebraic G -bundles on C w/ algebraic connection (flat is automatic since we are over a curve).

Aside If $V \xrightarrow{P} C$ is a principal G -bundle on C , we can look at

$$0 \rightarrow T_P \rightarrow T_{\text{Tot}(V)} \xrightarrow{dP} P^* T_C \rightarrow 0$$

and the direct image

$$0 \rightarrow P_* T_P \rightarrow P_* T_{\text{Tot}(V)} \rightarrow P_* P^* T_C \rightarrow 0$$

because G is an affine group.

pass to G -invariants

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (P \times^G T_P) & \longrightarrow & (P_* \otimes^G T_{\text{Tot}(V)}) & \longrightarrow & (P_* P^* T_C) \longrightarrow 0 \\
 & & \parallel & & \parallel & & \parallel \\
 & & V \times^G \mathfrak{g} & & \mathcal{A}(V) & & T_C \\
 & & & & \text{Atiyah algebra} & &
 \end{array} \quad (2)$$

This is the Atiyah sequence of V :

$$0 \longrightarrow \text{ad } V \longrightarrow \mathcal{A}(V) \longrightarrow T_C \longrightarrow 0$$

An algebraic connection on V is a splitting of this sequence:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{ad } V & \longrightarrow & \mathcal{A}(V) & \longrightarrow & T_C \longrightarrow 0 \\
 & & & & \swarrow & & \\
 & & & & \nabla & &
 \end{array}$$

∇ is flat if it is a map of \mathbb{A}^1 sheaves of Lie algebras.

Geometric properties of these moduli spaces:

- 1) \mathcal{Bun} is a smooth algebraic stack. If G is semisimple, then $\dim \mathcal{Bun} = (\dim G)(g-1)$.
- 2) \mathcal{Bun} is not quasi-compact and is not of finite type.
- 3) \mathcal{Bun}^{ss} = substack of semi-stable G -bundles
is open and dense and quasi-compact if we fix c_1 .

Note: $\pi_0 \mathcal{Bun}$ is coarsely representable by a pt.

$\pi_0 \mathcal{Bun}_{c_1=a}^{ss}$ " " " a projective variety (when G semi-simple
 $\dim \pi_0 \mathcal{Bun}_{c_1=a}^{ss} = (\dim G)(g-1)$).

4) \mathcal{Z}_{loc} is an algebraic stack of finite type and $\textcircled{3}$ quasi-compact.

Note: If $(v, \nabla) \in \mathcal{Z}_{loc}$, then the Atiyah sequence splits

$$0 \rightarrow \text{ad}v \rightarrow \mathcal{A}(v) \rightarrow T_c \rightarrow 0$$

$$\text{Ext}^1(T_c, \text{ad}v) = H^1(C, \text{ad}v \otimes \Omega_c^1) \xrightarrow{\text{tr}} H^1(C, \Omega_c^1)$$

\uparrow
 $c_1(v) = \text{tr}(\mathcal{A}(v))$

If G -semi-simple, then $\dim \mathcal{Z}_{loc} = 2 \cdot \dim G \cdot (g-1)$

Hint: If G reductive $\dim \mathcal{B}_{\text{un}} = \dim G(g-1) + \dim Z(G)$
 $= \dim G_{\text{ad}}(g-1) + (\dim Z(G) \cdot g)$

5) \mathcal{Z}_{loc} is not smooth in general. However, it is

~~HAZ~~ lci

Remark (v, ∇) is the same thing as a homomorphism

$$\text{-sm } \Pi_1(C, \pi) \rightarrow G$$

RH correspondence

$$\mathcal{Z}_{loc} \xrightarrow{\sim} \text{stack of reps. } \Pi_1(C, \pi) \rightarrow G$$

\uparrow analytic isomorphism
 \uparrow
 $(v, \nabla) \mapsto \text{monodromy}_\pi(\nabla) \xrightarrow{\text{ii}} \mathcal{M}_B(C, G)$

\mathcal{Z}_{loc} and $\mathcal{M}_B(C, G)$ are both alg. stacks of finite type

Note: $\mathcal{M}_B(C, G)$ does not depend on C as an alg. stack

\mathcal{Z}_{loc} does depend on C " "

$\mathcal{M}_g \rightarrow \text{moduli of } \mathcal{Z}_{loc}, \mathcal{C} \mapsto \mathcal{Z}_{loc}(\mathcal{C})$ is injective

First approximation to the geometric Langlands conjecture ④

(Dingfeng's best hope)

There is a canonical equivalence of categories

$$D_{\text{quoh}}(\text{Loc}, \mathcal{O}) \xrightarrow[\mathcal{C}]{\sim} D(\text{Bun}, \mathcal{D})$$

s.t. \mathcal{C} intertwines the natural 'symmetries' on both sides. (this naive formulation is false the categories on each side have different structure: compact objects...)

'symmetries' of $D_{\text{quoh}}(\text{Loc}, \mathcal{O})$:

Tensorization (Wilson operators): these are labelled by $n \in \mathbb{C}$, $\rho \in \text{Hom}(G, \text{GL}(E)) \leftrightarrow \rho^\mu$, $\mu \in \text{char}^+(G)$

Note: we have a universal bundle w/ connection

$$(\mathcal{V}, \nabla) \longrightarrow \text{Loc} \times \mathbb{C}$$

Then \mathcal{V} defines $\forall n \in \mathbb{C}$ $\mathcal{V}_n = \mathcal{V}|_{\text{Loc} \times \{n\}}$ a principal G -bundle on Loc .

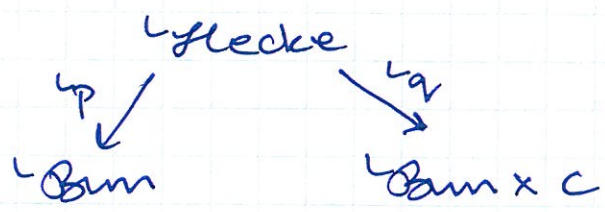
$\rho(\mathcal{V}_n) = \mathcal{V}_n \times^G E =$ associated vector bundle

The tensorization operator corresponding to (n, ρ)

$$\begin{aligned} \tilde{W}^{n, \rho} : D_{\text{quoh}}(\text{Loc}, \mathcal{O}) &\longrightarrow D_{\text{quoh}}(\text{Loc}, \mathcal{O}) \\ \mathcal{F} &\longmapsto \mathcal{F} \otimes \rho(\mathcal{V}_n) \end{aligned}$$

symmetries of $D(\text{Bun}, \mathcal{D})$

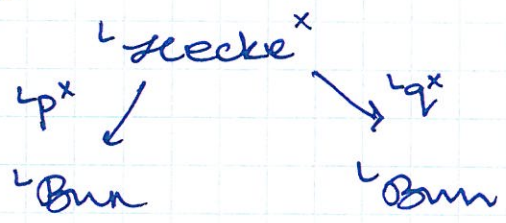
Hecke (t) twist operators. Labelled by $n \in \mathbb{C}$ and $\rho \in \text{Hom}(G, \text{GL}(E))$



Hecke = stack of $(r, r', \alpha, \beta: r|_{\mathbb{C}-\{n\}} \xrightarrow{\sim} r'|_{\mathbb{C}-\{n\}})$
 p, q obvious projection maps

Hecke is not an algebraic stack, but it is ind-algebraic.

fix $x \in \mathbb{C}$ get substack



Hecke^x is a locally trivial fibration over Bun w/ fibres $\cong \text{Gr}_g$

Remark Hecke^x is formally smooth but not smooth as an ind-stack.

Gr_g is formally smooth but not smooth as an ind-scheme. This follows from $G(t)$ not being smooth as an ind-scheme.

Aside An ind-scheme is $\varinjlim_{a \in A} X_a =: X$

X_a - quasi-compact schemes
 $X_a \hookrightarrow X_b$ for $a \leq b$
 \uparrow
 closed immersions

X is called smooth at a pt if it can locally near that pt as $\varinjlim U_a$

X is formally smooth if it satisfies infinitesimal lifting property.

U_a - smooth

Thm (Simpson - Teleman) $G(t)$ is not smooth! (6)

Thm (Simpson - Teleman) If X is ind-projective which is smooth at each pt, then the Hodge to de Rham spectral sequence on X degenerates at E_1 .

Thm (Greiffisher - Grejnowski - Teleman)

$$H_{\text{DR}}^2(G(t), \mathbb{C}) = \mathbb{C}$$

$$H_{\text{DR}}^1(G(t), \Omega^1_{G(t)}) = H^1(g(t), g; \int_{\mathcal{G}} g[[t]] dt)$$