Introduction to Geometric Langlands I

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Formulation of the geometric Langlands conjecture (this is the goal)

Background data:

C - smooth projective curve$/\mathbb{C}$, genus $g > 1$

G - complex reductive group

$T \subset B \subset G$; $\mathrm{char}(T) = \mathrm{Hom}(T, \mathbb{C}^*)$, $\mathrm{cochar}(T) = \mathrm{Hom}(\mathbb{C}^*, T)$

$\mathrm{root}_{\mathfrak{g}} \subset \mathrm{char}(T) \subset \mathrm{weight}_{\mathfrak{g}}$

$\mathrm{coweight}_{\mathfrak{g}} \subset \mathrm{cochar}(T) \subset \mathrm{coroot}_{\mathfrak{g}}$

Two reductive groups, G, ${}^LG$ are Langlands dual if $\mathrm{char}(G) \cong \mathrm{cochar}({}^LG)$

moduli objects: $\mathcal{B}un$ = moduli stack of principle algebraic G-bundles on C

$\mathcal{L}oc$ = moduli stack of principle algebraic G-bundles on C w/ algebraic connection (flat is auto-matic since we are over a curve).

Aside If $V \xrightarrow{p} C$ is a principal G-bundle on C, we can look at

$$0 \to T_p \to T_{Tot(V)} \xrightarrow{dp} p^*T_C \to 0$$

and the direct image

$$0 \to p_*T_p \to p_*T_{Tot(V)} \to \underbrace{p_*p^*T_C}_{\text{because } G \text{ is an } \underline{\underline{\text{affine}}} \text{ group.}} \to 0$$

pass to G-invariants

$$0 \to (p_* T_p)^G \to (p_* T_{\mathrm{Tot}(V)})^G \to (p_* p^* T_C)^G \to 0$$

$$(p_* T_p)^G = V \times_{ad} \mathfrak{g}, \quad (p_* T_{\mathrm{Tot}(V)})^G = \mathcal{A}(V), \quad (p_* p^* T_C)^G = T_C$$

Atiyah algebra

This is the Atiyah sequence of V:

$$0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \to T_C \to 0$$

An algebraic connection on V is a splitting of this sequence:

$$0 \to \mathrm{ad}\, V \to \mathcal{A}(V) \underset{\nabla}{\leftrightarrows} T_C \to 0$$

∇ is flat if it is a map of sheaves of Lie algebras.

Geometric properties of these moduli spaces:

1) $\mathcal{B}un$ is a smooth algebraic stack. If G is semisimple, then $\dim \mathcal{B}un = (\dim G)(g-1)$.

2) $\mathcal{B}un$ is not quasi-compact and is not of finite type.

3) $\mathcal{B}un^{ss}$ = substack of semi-stable G-bundles is open $\underbrace{\text{and dense}}$ and quasi-compact if we fix c_1

Note: $\pi_0 \mathcal{B}un$ is coarsely representable by a pt.

$\pi_0 \mathcal{B}un^{ss}_{c_1=a}$ " " " a projective variety (when G semi-simple $\dim \pi_0 \mathcal{B}un^{ss}_{c_1=a} = (\dim G)(g-1)$.).

4) $\mathcal{L}oc$ is an algebraic stack of finite type and quasi-compact.

<u>Note</u>: If $(V, \nabla) \in \mathcal{L}oc$, then the Atiyah sequence splits $(0 \to \operatorname{ad} V \to \mathcal{A}(V) \to T_C \to 0$

$$\operatorname{Ext}^1(T_C, \operatorname{ad} V) = H^1(C, \operatorname{ad} V \otimes \Omega^1_C)$$

$$-c_1(V) = \operatorname{tr}(\mathcal{A}(V)) \qquad \downarrow \operatorname{tr} \qquad H^1(\Omega^1_C))$$

If G - semi-simple, then $\dim \mathcal{L}oc = 2 \cdot \dim G \cdot (g-1)$

<u>Hint</u>: If G reductive $\dim \mathcal{B}un = \dim G(g-1) + \dim Z(G)$

$$= \dim G_{ad}(g-1) + (\dim Z(G) \cdot g)$$

5) $\mathcal{L}oc$ is not smooth in general. However, it is ~~[illegible]~~ lci

<u>Remark</u> (V, ∇) is the same thing as a homomorphism $\pi_1(C, x) \to G$

<u>RH correspondence</u>

$$\mathcal{L}oc^{an} \xrightarrow{\sim} \text{stack of reps. } \pi_1(C, x) \to G$$

analytic isomorphism

$$(V, \nabla) \longmapsto \text{monodromy}_x(\nabla) \qquad \mathcal{M}_B(C, G)$$

$\mathcal{L}oc$ and $\mathcal{M}_B(C, G)$ are both alg. stacks of finite type

<u>Note</u>: $\mathcal{M}_B(C, G)$ does <u>not</u> depend on C as an alg. stack

$\mathcal{L}oc$ does depend on C " "

$\mathcal{M}_g \to$ moduli of $\mathcal{L}oc$, $C \mapsto \mathcal{L}oc(C)$ is injective

First approximation to the Geometric Langlands conjecture

(Drinfeld's best hope)

There is a canonical equivalence of categories

$$D_{qcoh}(\mathcal{Loc}, \mathcal{O}) \xrightarrow[C]{\sim} D({}^L\mathcal{B}un, \mathcal{D})$$

s.t C 'intertwines the natural 'symmetries' on both sides. (this naive formulation is false the categories on each side have different structure: compact objects . . .)

'symmetries' of $D_{qcoh}(\mathcal{Loc}, \mathcal{O})$:

Tensorization (Wilson operators): these are labelled by $x \in C$, $\rho \in Hom(G, GL(E)) \leftrightarrow \rho^{\mu}$, $\mu \in char^{+}(G)$

Note: we have a universal bundle w/ connection

$$(\mathcal{V}, \nabla) \longrightarrow \mathcal{Loc} \times C$$

Then $\mathcal{V}$ defines $\mathcal{V}_x = \mathcal{V}|_{\mathcal{Loc} \times \{x\}}$ a principal G-bundle on $\mathcal{Loc}$.

$$\rho(\mathcal{V}_x) = \mathcal{V}_x \overset{G}{\times} E \quad = \quad \text{associated vector bundle}$$

The tensorization operator corresponding to (x, ρ) is $W^{x,\rho}: D_{qcoh}(\mathcal{Loc}, \mathcal{O}) \longrightarrow D_{qcoh}(\mathcal{Loc}, \mathcal{O})$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \rho(\mathcal{V}_x)$$

symmetries of $D({}^L\mathcal{B}un, \mathcal{D})$

Hecke ('t Hooft) operators. Labelled by $x \in C$ and $\rho \in Hom(G, GL(E))$

$$\begin{tikzcd} & {}^L\mathcal{H}ecke \arrow[dl, "{}^Lp"'] \arrow[dr, "{}^Lq"] & \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \times C \end{tikzcd}$$

${}^L\mathcal{H}ecke$ = stack of $(V, V', x, \beta : V|_{C-\{x\}} \xrightarrow{\sim} V'|_{C-\{x\}})$

${}^Lp, {}^Lq$ obvious projection maps

${}^L\mathcal{H}ecke$ is <u>not</u> an algebraic stack, but it is ind-algebraic.

Fix $x \in C$ get substack

$$\begin{tikzcd} & {}^L\mathcal{H}ecke^x \arrow[dl, "{}^Lp^x"'] \arrow[dr, "{}^Lq^x"] & \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \end{tikzcd}$$

${}^L\mathcal{H}ecke^x$ is a locally trivial fibration over ${}^L\mathcal{B}un$ w/ fibres $\cong Gr_G$

<u>Remark</u> ${}^L\mathcal{H}ecke^x$ is formally smooth but not smooth as an ind-stack.

$\rightsquigarrow$ Gr_G is formally smooth but not smooth as an ind-scheme. This follows from $G((t))$ not being smooth as an ind-scheme.

<u>Aside</u> An ind-scheme is $\varinjlim_{a \in A} X_a =: \mathcal{X}$

X_a - quasi-compact schemes

$X_a \hookrightarrow X_b$ for $a \leq b$

closed immersions

$\mathcal{X}$ is formally smooth if it satisfies infinitesimal lifting property.

$\mathcal{X}$ is called smooth at a pt if it can locally near that pt as $\varinjlim U_a$

U_a - smooth

Thm (Simpson - Teleman) $G((t))$ is not smooth!

Thm (Simpson - Teleman) If X is ind-projective which is smooth at each pt, then the Hodge to deRham spectral sequence on $\mathcal{X}$ degenerates at E_1.

Thm (~~Gpq~~ Fishel - Grojnowski - Teleman)

$$H^2_{dR}(G((t)), \mathbb{C}) = \mathbb{C}$$

$$H^1_{dR}(G((t)), \Omega^1_{G((t))}) = H^1(\mathfrak{g}((t)), \mathfrak{g}; \mathfrak{g}[[t]]\,dt)$$

$$\underset{\mathfrak{g}}{\cup}$$