

# Geometric Langlands (the classical limit) II

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①

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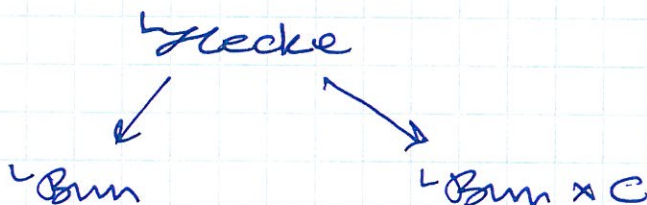
Recall best hope

$$c: D_{\text{quoh}}(\text{Loc}, \mathcal{O}) \xrightarrow{\sim} D(\mathcal{L}\text{-Bun}, \mathcal{D})$$

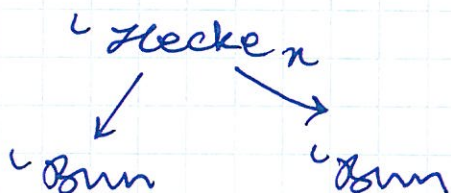
↑ infinitesimal natural symmetries

On LHS the natural symmetries were given by tensorization  $W^{n,p}(F) = F \otimes p(\mathcal{V}_n)$

RHS symmetries are Hecke operators. we had



and



Note There is a composition of correspondences (standard fibre product game)

Given  $n \in \mathbb{C}$ ,  $\rho \in \text{Rep}(G) = \text{char}^+(G)$

$$\mu = \text{char}^+(\mathcal{L}G)$$

$$\Rightarrow \mathcal{L}\text{-Hecke}_x^\rho \subset \mathcal{L}\text{-Hecke}_n$$

"  $\{(v, v', \beta) \text{ with poles of } \beta \text{ bounded by } \rho\}$

explicitly  $(v, v', \beta) \in \mathcal{L}\text{-Hecke}_x^\rho$  iff for every  $\lambda \in \text{char}^+(\mathcal{L}G)$

the induced map of locally free sheaves

$$\rho^\lambda(v) \xrightarrow{\rho^\lambda(\beta)} \rho^\lambda(v') \quad (\langle \lambda, \mu \rangle \times) \quad \text{for all } \lambda \quad (2)$$

${}^L\text{Hecke}_x^\rho$  is an algebraic stack

$${}^L\text{Hecke}_x = \varinjlim_{\rho} {}^L\text{Hecke}_x^\rho,$$

the projection maps  ${}^L\text{Hecke}_x^\rho \rightarrow {}^L\text{Bun}$

are locally (étale) trivial fibrations, the fibres are closures of affine schubert varieties.

Note:  ${}^L\text{Hecke}_x^\rho$  is smooth if and only if  $\mu$  is minuscule

The Hecke operator given by  $(n, \rho)$  is the functor

$${}^L\mathcal{H}^{n, \rho}: \mathcal{D}({}^L\text{Bun}, \mathcal{O}) \rightarrow \mathcal{G}$$

$$F \mapsto \mathcal{L}_{\mathbb{G}_m}^\rho \left( \mathcal{L}_{\rho^\lambda}^\rho \otimes \mathcal{I}C_{\text{Hecke}_x^\rho} \right)$$

Best hope requires

$$c \circ W^{n, \rho} = {}^L\mathcal{H}^{n, \rho} \circ c \quad \text{for all } n, \rho$$

$\mathcal{D}_{\text{quasi}}(\text{Loc}, \mathcal{O})$  has an orthogonal collection of spanning objects: structure sheaves of pts.

This is a basis of eigensheaves for  $W^{n, \rho}$

If  $V = (v, \nabla)$  is a  $\mathbb{G}$ -local system, then

$c(\mathcal{O}_V)$  will have to be an eigen  $\mathcal{D}$ -module for the action of  ${}^L\mathcal{H}^{n, \rho}$ . i.e.,  ${}^L\mathcal{H}^{n, \rho}(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \otimes \rho(v_x)$



If we don't fix  $n$ , then we still have "checked" ③

$$\text{and } {}^L H^p: D(\mathcal{B}un, \mathcal{D}) \longrightarrow D({}^L \mathcal{B}un \times c, \mathcal{D})$$

$$\text{and } {}^L H^p(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \boxtimes p(V)$$

Best hope is true for split tori  $\leftarrow$  w/ just a small correction

What is wrong w/ the Best hope?

The categories are just the wrong sizes!

$\mathcal{D}_{\text{quasi}}(\mathcal{Z}_{\text{oc}}, \mathcal{O})$  has an orthogonal decomposition

$$\Pi_0(\mathcal{Z}_{\text{oc}}) = H^2(c_{\mathcal{B}}, \Pi_1(G)_{\text{tor}}) \simeq \Pi_1(G)_{\text{tor}}$$

$$\leadsto \mathcal{Z}_{\text{oc}} = \coprod_{\gamma \in \Pi_1(G)_{\text{tor}}} \mathcal{Z}_{\text{oc}, \gamma}$$

$\mathcal{Z}_{\text{oc}}$  is a stack s.t. for each object  $V = (V, \nabla)$

$Z(G) \subset \text{Aut}(V)$ . There is a rigidification of  $\mathcal{Z}_{\text{oc}}$ :

$$\underline{\mathcal{Z}_{\text{oc}}} = \mathcal{Z}_{\text{oc}} \! / \! /_{Z(G)}$$

$\uparrow$   
algebraic stack w/ a map  $\mathcal{Z}_{\text{oc}} \rightarrow \underline{\mathcal{Z}_{\text{oc}}}$  so that

$\mathcal{Z}_{\text{oc}}$  is  $Z_0(G)$  is a gerbe over  $\underline{\mathcal{Z}_{\text{oc}}}$

$\mathcal{D}_{\text{quasi}}(\underline{\mathcal{Z}_{\text{oc}}}, \mathcal{O})$  is irreducible.

Remark If  $\mathcal{X}$  is an algebraic stack, it comes equipped w/ a

group scheme  $I_{\mathcal{X}} \rightarrow \mathcal{X}$ . Suppose we have a flat subgroup

$$\begin{array}{c} \mathcal{X} \times_{\Delta} \mathcal{X} \\ \uparrow \Delta \\ \mathcal{X} \times \mathcal{X} \end{array}$$

scheme  $H < I_{\mathcal{X}}$  ( $\Rightarrow H$  is normal in  $I_{\mathcal{X}}$ )

then we can rigidify  $\mathcal{X}$  by  $H$

i.e. construct a new stack  $\mathcal{X} \! / \! /_H$  s.t.:



1)  $\mathcal{K} \xrightarrow{p} \mathcal{K} // H$

locally on  $\mathcal{K} // H$

2)  $p^* I_{\mathcal{K} // H} = I_{\mathcal{K} / H}$

$\mathcal{K} \cong \mathcal{K} // H \times BH$

3)  $p: \mathcal{K} \rightarrow \mathcal{K} // H$  is a flat  $H$ -gerbe

Example

$\mathcal{K} =$  stack of rank  $n$  vector bundles on  $C$

$Z(GL_n(\mathbb{C})) = \mathbb{C}^*$ ;  $\mathbb{C}^* \subset I_{\mathcal{K}}$  we can

rigidify  $\mathcal{K} // \mathbb{C}^*$ .  $\mathcal{K} // \mathbb{C}^*$  is the moduli stack s.t.

$(\mathcal{K} // \mathbb{C}^*)(S)$  has

objects:  $E \rightarrow S \times C$  flat family of rank  $n$ -vector bundles

morphisms:  $(L, \psi) \quad L \rightarrow S$   
line bundle

Remark if  $\mathcal{Y}$ -stack  $\mathcal{X} \rightarrow \mathcal{Y} \xrightarrow{H} \mathcal{Y}$  sheaf of abelian groups  $\mathcal{X} \rightarrow \mathcal{Y}$  is an  $H$ -gerbe,

then  $D_{\text{quoh}}(\mathcal{X}, \mathcal{O}) = \bigsqcup_{\alpha \in \text{char } H} D_{\text{quoh}}(\mathcal{X}, \mathcal{O})^\alpha$   
 $\parallel$   
 $D_{\text{quoh}}(\mathcal{Y}, \mathcal{O}, \alpha)$

Conclusion

$D_{\text{quoh}}(\mathcal{Z}_{\text{oc}}, \mathcal{O}) = \bigsqcup_{(\sigma, \alpha) \in \pi_1(G)_{\text{tor}} \times Z(G)^\wedge} D_{\text{quoh}}(\mathcal{Z}_{\text{oc}, \sigma}, \mathcal{O}; \alpha)$

What about  $D(\mathcal{L}\text{Bun}, \mathcal{D})$  ?

(5)

$\mathcal{L}\text{Bun}$  can be stratified  $\mathcal{L}\text{Bun} = \mathcal{L}\text{Bun} // Z(\mathcal{L}G)$

and we have connected components

$$\mathcal{L}\text{Bun} = \bigsqcup_{\alpha \in \pi_0(\mathcal{L}\text{Bun})} \mathcal{L}\text{Bun}_\alpha$$

$$\pi_0(\mathcal{L}\text{Bun}) = H^2(C, \pi_1(\mathcal{L}G))$$

$\mathcal{L}\text{Bun} \rightarrow \underline{\mathcal{L}\text{Bun}}$  is  $\mathbb{Q} \simeq Z(\mathcal{L}G)$ -gerbe

This factors as

$$\begin{array}{ccc} \mathcal{L}\text{Bun} & \longrightarrow & \mathcal{L}\text{Bun} // Z_0(\mathcal{L}G) = \mathcal{L}\text{Bun} \\ & & \downarrow \\ & & \mathcal{L}\text{Bun} // \pi_1(Z(\mathcal{L}G)) \\ & & \parallel \\ & & \underline{\mathcal{L}\text{Bun}} \end{array} \quad \begin{array}{c} \uparrow \\ \text{(no script)} \end{array}$$

Note  $\pi_1(\mathcal{L}G) = Z(G)^\wedge$  ;  $Z_0(\mathcal{L}G) = (\pi_1(G)_{\text{free}})^\wedge$   
 $\pi_0(Z(\mathcal{L}G)) = (\pi_1(G)_{\text{tor}})^\wedge$

$\leadsto$  This indicates that RHS of the best hope one should take  $\mathcal{L}\text{Bun} = \mathcal{L}\text{Bun} // Z_0(\mathcal{L}G)$ ; Then

$$D(\mathcal{L}\text{Bun}, \mathcal{D}) = \bigsqcup_{(\alpha, r) \in \pi_1(\mathcal{L}G) \times \pi_0(Z(\mathcal{L}G))} D(\mathcal{L}\text{Bun}_\alpha, \mathcal{D}; r)$$



(6)

Example  $G = GL_1$ ,

$$\mathcal{L}oc_1 = \underline{\mathcal{L}oc}_1 \times BC^* = T_0^v J^0(C) \times BC^*$$

$$\mathcal{P}un_1 = \text{Pic}(C) = J^0(C) \times \mathbb{Z} \times BC^*$$

$$R\mathcal{L}oc_1 = (T_0^v J^0(C), \mathcal{O} \otimes A) \times BC^*$$

↑  
freely zend. DGA w/ 1 gen in deg -1.

$$= T_0^v J^0(C) \times BC^* \times \text{Rspec}(A)$$

$$c: D_{\text{quoh}}(T_0^v J^0(C), \mathcal{O}) \xrightarrow{\sim} D(J^0(C), \mathcal{O})$$

Laumon Fourier transform

$$D_{\text{quoh}}(BC^*, \mathcal{O}) \xrightarrow{\sim} D(\mathbb{Z}, \mathcal{D})$$

$$D_{\text{quoh}}(\text{Rspec } A, \mathcal{O}) \xrightarrow{\sim} D(BC^*, \mathcal{D})$$