# Geometric Langlands (the classical limit) II (T. Panter)

(Freiburg, 31 Jul 2012)

Recall best hope

$$c : D_{qcoh}(\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathcal{B}un, \mathcal{D})$$

↑ intertwines natural symmetries

On LHS the natural symmetries were given by tensorization $W^{x,\rho}(F) = F \otimes \rho(\mathcal{V}_x)$

RHS symmetries are Hecke operators. We had

$$\begin{array}{ccc} & {}^L\mathcal{H}ecke & \\ \swarrow & & \searrow \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \times C \end{array}$$

and

$$\begin{array}{ccc} & {}^L\mathcal{H}ecke_x & \\ \swarrow & & \searrow \\ {}^L\mathcal{B}un & & {}^L\mathcal{B}un \end{array}$$

<u>Note</u> There is a composition of correspondences (standard fibre product game)

Given $x \in C$, $\rho \in Rep(G) = char^+(G)$
$\leftrightarrow \mu = cochar^+({}^LG)$

$\rightsquigarrow$ ${}^L\mathcal{H}ecke_x^\rho \subset {}^L\mathcal{H}ecke_x$
$\quad\quad \| $
$\{(V, V', \beta)$ with poles of $\beta$ bounded by $\rho\}$

explicitly $(V, V', \beta) \in {}^L\mathcal{H}ecke_x^\rho$ iff for every $\lambda \in char^+({}^LG)$ the induced map of locally free sheaves

$$\rho^\lambda(V) \xrightarrow{\rho^\lambda(\beta)} \rho^\lambda(V')(\langle\lambda,\mu\rangle x) \quad \text{for all } \lambda$$

${}^L\mathcal{H}ecke^\rho_x$ is an algebraic stack

$${}^L\mathcal{H}ecke_x = \varinjlim_\rho {}^L\mathcal{H}ecke^\rho_x,$$

the projection maps ${}^L\mathcal{H}ecke^\rho_x \to {}^L\mathcal{B}un$ are locally (étale) trivial fibrations, the fibres are closures of affine schubert varieties.

<u>Note</u>: ${}^L\mathcal{H}ecke^\rho_x$ is smooth if <u>and only if</u> $\mu$ is miniscule

The Hecke operator given by $(x,\rho)$ is the functor

$${}^LH^{x,\rho} : D({}^L\mathcal{B}un, \mathcal{D}) \circlearrowleft$$

$$F \longmapsto {}^Lq^\rho_{x!}\left({}^Lp^{\rho *}_x \mathcal{F} \otimes IC_{\mathcal{H}ecke^\rho_x}\right)$$

Best hope requires

$$c \circ W^{\lambda,\rho} = {}^LH^{x,\rho} \circ c \qquad \text{for all } x, \rho$$

$D_{qcoh}(\mathcal{L}oc, \mathcal{O})$ has an orthogonal collection of spanning objects: structure sheaves of pts.

This is a basis of eigensheaves for $W^{x,\rho}$

If $V = (V, \nabla)$ is a G-local system, then $c(\mathcal{O}_V)$ will have to be an eigen $\mathcal{D}$-module for the action of ${}^LH^{x,\rho}$. i.e., ${}^LH^{\rho,x}(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \otimes \rho(V_x)$

If we don't fix $x$, then we still have "Hecke"

and ${}^L H^P: D({}^L\mathrm{Bun}, \mathcal{D}) \longrightarrow D({}^L\mathrm{Bun} \times C, \mathcal{D})$

and ${}^L H^P(c(\mathcal{O}_V)) = c(\mathcal{O}_V) \boxtimes \rho(V)$

$\boxed{\text{Best hope is true for split tori}}$ — w/ just a small correction

What is wrong w/ the Best hope?

The categories are just the wrong sizes! on

& $D_{qcoh}(\mathcal{L}oc, \mathcal{O})$ has an orthogonal decomposition

$\pi_0(\mathcal{L}oc) = H^2(C, \pi_1(G)_{tor}) \simeq \pi_1(G)_{tor}$

$\rightsquigarrow \mathcal{L}oc = \bigsqcup_{\gamma \in \pi_1(G)_{tor}} \mathcal{L}oc_\gamma$

$\mathcal{L}oc$ is a stack s.t. for each object $V = (V, \nabla)$

$Z(G) \subset \mathrm{Aut}(V)$. There is a rigidification of $\mathcal{L}oc$:

$\underline{\mathcal{L}oc} = \mathcal{L}oc \,/\!\!/\, Z_0(G)$

$\uparrow$

algebraic stack w/ a map $\mathcal{L}oc \longrightarrow \underline{\mathcal{L}oc}$ so that

$\mathcal{L}oc$ is $Z_0(G)$ is a gerbe over $\underline{\mathcal{L}oc}$

$D_{qcoh}(\underline{\mathcal{L}oc}_\gamma, \mathcal{O})$ is irreducible.

<u>Remark</u> If $\mathcal{X}$ is an algebraic stack, it comes equipped w/ a group scheme $I_{\mathcal{X}} \longrightarrow \mathcal{X}$. Suppose we have a flat subgroup

$$I_{\mathcal{X}} = \mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$$

scheme $H < I_{\mathcal{X}}$ ($\Rightarrow H$ is normal in $I_{\mathcal{X}}$)

then we can rigidify $\mathcal{X}$ by $H$

i.e. construct a new stack $\mathcal{X} \,/\!\!/\, H$ s.t.:

1) $\mathcal{X} \xrightarrow{p} \mathcal{X} /\!\!/ H$

2) $p^* I_{\mathcal{X}/\!\!/ H} = I_{\mathcal{X}}/H$

locally on $\mathcal{X}/\!\!/H$
$\mathcal{X} \simeq \mathcal{X}/\!\!/H \times BH$

3) $p : \mathcal{X} \to \mathcal{X}/\!\!/H$ is a flat $H$-gerbe

Example

$\mathcal{X}$ = stack of rank $n$ vector bundles on $C$

$Z(GL_n(\mathbb{C})) = \mathbb{C}^*$; $\mathbb{C}^* \subset I_{\mathcal{X}}$ we can rigidify $\mathcal{X} /\!\!/ \mathbb{C}^*$. $\mathcal{X}/\!\!/\mathbb{C}^*$ is the moduli stack s.t.

$(\mathcal{X}/\!\!/\mathbb{C}^*)(S)$ has

objects: $E \to S \times C$ flat family of rank $n$-vector bundles

morphisms: $(L, \varphi)$ $L \to S$ line bundle

Remark If $\mathcal{Y}$-stack ~~$\mathcal{X}$~~ $\to \mathcal{Y}$ $H \to \mathcal{Y}$ sheaf of abelian groups $\mathcal{X} \to \mathcal{Y}$ is an $H$-gerbe, then

$$D_{qcoh}(\mathcal{X}, \mathcal{O}) = \bigsqcup_{\alpha \in \mathrm{Char}\, H} D_{qcoh}(\mathcal{X}, \mathcal{O})^{\alpha}$$

$$D_{qcoh}(\mathcal{X}, \mathcal{O})^{\alpha} = D_{qcoh}(\mathcal{Y}, \mathcal{O}, \alpha)$$

Conclusion

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) = \bigsqcup_{(\gamma, \alpha) \in \pi_1(G)_{tor} \times Z(G)^{\wedge}} D_{qcoh}(\mathcal{L}oc_{\gamma}, \mathcal{O}; \alpha)$$

What about $D({}^L\mathbb{B}\mathrm{un}, \mathcal{D})$?

${}^L\mathbb{B}\mathrm{un}$ can be ~~r~~rigidified ${}^L\underline{\mathrm{Bun}} = {}^L\mathbb{B}\mathrm{un} /\!\!/ Z({}^LG)$

and we have connected components

$$ {}^L\mathbb{B}\mathrm{un} = \coprod_{\alpha \in \pi_0({}^L\mathbb{B}\mathrm{un})} {}^L\mathbb{B}\mathrm{un}_\alpha $$

$\pi_0({}^L\mathbb{B}\mathrm{un}) = H^2(C, \pi_1({}^LG))$

${}^L\mathbb{B}\mathrm{un} \longrightarrow {}^L\underline{\mathrm{Bun}}$ is ~~a~~ a $Z({}^LG)$-gerbe

This factors as

$$\begin{tikzcd}
{}^L\mathbb{B}\mathrm{un} \arrow[r] & {}^L\mathbb{B}\mathrm{un} /\!\!/ Z_0({}^LG) = {}^L\mathrm{Bun} \arrow[d] \\
& {}^L\mathbb{B}\mathrm{un} /\!\!/ \pi_1(Z({}^LG)) = {}^L\underline{\mathrm{Bun}}
\end{tikzcd}$$

$\uparrow$ (no script)

<u>Note</u> $\pi_1({}^LG) = Z(G)^\wedge$ ; $Z_0({}^LG) = (\pi_1(G)_{\cancel{tor}\ free})^\wedge$

$\pi_0(Z({}^LG)) = (\pi_1(G)_{tor})^\wedge$

$\rightsquigarrow$ This indicates that RHS of the best hope one should take ${}^L\mathrm{Bun} = {}^L\mathbb{B}\mathrm{un} /\!\!/ Z_0({}^LG)$ ; Then

$$ D({}^L\mathrm{Bun}, \mathcal{D}) = \coprod_{(\alpha, \gamma) \in \pi_1({}^LG) \times \pi_0(Z({}^LG))} D({}^L\underline{\mathrm{Bun}}_\alpha, \mathcal{D}; \gamma) $$

<u>Example</u> $G = GL_1$,

$\mathcal{L}oc_1 = \underline{\mathcal{L}oc}_1 \times B\mathbb{C}^* = T^\vee_\theta J^0(C) \times B\mathbb{C}^*$

$\mathcal{B}un_1 = \mathrm{Pic}(C) = J^0(C) \times \mathbb{Z} \times B\mathbb{C}^*$

$R\mathcal{L}oc_1 = (T^\vee_\theta J^0(C), \mathcal{O} \otimes A) \times B\mathbb{C}^*$

$\uparrow$ freely gen'd. DGA w/ 1 gen in deg $-1$.

$= T^\vee_\theta J^0(C) \times B\mathbb{C}^* \times R\mathrm{spec}(A)$

$c: D_{qcoh}(T^\vee_\theta J_0(C), \mathcal{O}) \xrightarrow{\sim} D(J^0(C), \mathcal{O})$

Laumon Fourier transform

$D_{qcoh}(B\mathbb{C}^*, \mathcal{O}) \xrightarrow{\sim} D(\mathbb{Z}, \mathcal{D})$

$D_{qcoh}(R\mathrm{spec}\, A, \mathcal{O}) \xrightarrow{\sim} D(B\mathbb{C}^*, \mathcal{D})$