Classical limit of ~~the~~ geometric Langlands III (T. Panter)

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Recall:

$$D_{qcoh}(\mathcal{L}oc, \mathcal{O}) = \coprod_{(\gamma,\alpha)\in\pi_1(G)_{tor}\times Z(G)^\wedge} D_{qcoh}(\underline{\mathcal{L}oc}_\gamma, \mathcal{O};\alpha)$$

$$D({}^L\mathcal{B}un, \mathcal{D}) = \coprod_{(\alpha,\gamma)\in\pi_1({}^LG)\times\pi_0(Z({}^LG))^\wedge} D({}^L\underline{\mathcal{B}un}_\alpha, \mathcal{D};\gamma)$$

$$D({}^L\mathrm{Bun}, \mathcal{D}) = \coprod_{\alpha,\gamma} D({}^L\mathrm{Bun}_\alpha, \mathcal{D};\gamma)$$

$${}^L\mathcal{B}un /\!\!/ Z_0({}^LG)$$

Computations for GL_1 suggests that the geometric Langlands correspondence should identify $D_{qcoh}(\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun}, \mathcal{D})$

or $\quad D_{qcoh}(R\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathcal{B}un, \mathcal{D})$

+ should identify pieces labelled by the same data.

This modification also ends up not being enough, since $D({}^L\mathrm{Bun}, \mathcal{D})$ behaves as the derived category of quasi-coherent sheaves on a smooth space. The other side doesn't.

<u>Aside</u>

smoothness is a categorical notion

<u>Def</u> If $\mathcal{C}$ is a ∞-complete dg-category, then $X \in Ob(\mathcal{C})$ is called <u>compact</u> if $Hom_{\mathcal{C}}(X, -)$ commutes w/ colimits

The full subcategory of compact objects in $\mathcal{C}$ is denoted $Perf(\mathcal{C})$.

$\mathcal{C}$ is said to be compactly generated if $\mathcal{C} \simeq \widehat{Perf(\mathcal{C})}$.

<u>Thm</u> (~~Nem~~ Neeman, TT) X - quasi compact, quasi separated scheme.

$$Perf(D_{qcoh}(X)) = Perf(X)$$

complexes locally qis to complexes of vector bundles

and $D_{qcoh}(X) = \widehat{Perf}(X)$

<u>Fact</u> X is smooth iff $Perf(X) = D_{coh}(X)$

X is singular iff

$$Perf(X) \subsetneq D_{coh}(X)$$

<u>NC - smoothness</u> $\mathcal{C}$ - complete + cocomplete.

We say that $\mathcal{C}$ is NC-smooth iff $\mathcal{C}$ is compact viewed as an object in the category $\widehat{\mathcal{C} \otimes \mathcal{C}}^{op}$

<u>Note</u> X is smooth iff $\widehat{Perf(X)}$ is nc-smooth

If X is singular, it can happen that $\widehat{D_{coh}}(X)$ is nc-smooth.

Def If X is a scheme/stack,

$\widehat{D_{coh}}(X)$ = stable derived category of X

$\|$

$\mathrm{Ind\,Coh}(X)$ = category of ind-coherent sheaves.

$\uparrow$

Explicit model = unbounded complexes of injective (morphisms = homotopy classes of maps).

Have a localization functor

$$\mathrm{Ind\,Coh}(X) \longrightarrow D_{qcoh}(X)$$

which has a right adjoint

$$\exists : D_{qcoh}(X) \hookrightarrow \mathrm{Ind\,Coh}(X).$$

Have a sequence

$$\begin{array}{ccccccc} \mathrm{Perf}(X) & \subset & D_{coh}(X) & \subset & D_{qcoh}(X) & \subset & \mathrm{Ind\,Coh}(X) \\ & & & & \| & & \| \\ & & & & \widehat{\mathrm{Perf}}(X) & & \widehat{D_{coh}}(X) \end{array}$$

Derived category of singularities:

$$D_{sing}(X) := D_{coh}(X)/\mathrm{Perf}(X)$$

$$\widehat{D_{sing}}(X) = \mathrm{Ind\,Coh}(X)/D_{qcoh}(X)$$

Facts

- If U is a quasi-compact scheme/stack, then $D(U, \mathcal{D})$ is compactly generated

- (Arkhipov - Gaitsgory) ${}^L\mathrm{Bun} = \bigcup_\alpha U_\alpha$ such that
quasicompact

$$j_{\alpha\beta} : U_\alpha \hookrightarrow U_\beta$$

satisfy $j_{\alpha\beta *} : D(U_\alpha, \mathcal{D}) \longrightarrow D(U_\beta, \mathcal{D})$ preserve perfectness

- (Drinfeld - Gaitsgory) If U is a quasi-compact stack, then

$$\mathrm{Perf}(D(U, \mathcal{D})) = D_{coh}(U, \mathcal{D}).$$

Moreover,

$$\mathrm{Perf}(D({}^L\mathrm{Bun}, \mathcal{D})) = \text{!-extensions of compact objects in } D(U_\alpha, \mathcal{D})$$

$\Rightarrow D({}^L\mathrm{Bun}, \mathcal{D})$ is compactly generated.

Remark Due to the above we should probably extend $D_{qcoh}(\mathcal{L}oc, \mathcal{O})$ to $\mathrm{IndCoh}(\mathcal{L}oc, \mathcal{O})$. However, $\mathrm{IndCoh}(\mathcal{L}oc, \mathcal{O})$ is too big.

Reason: $c : \mathrm{IndCoh}(\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun}, \mathcal{D})$ should intertwine tensorization and Hecke operators but should also be functorial in G.

<u>Functoriality in G</u>:

If $P \subset G$ is parabolic $\longleftrightarrow$ ${}^LP \subset {}^LG$
$M = \text{Levi}(P)$ ${}^LM = \text{Levi}({}^LP)$

get:

$$
\begin{tikzcd}
 & \mathcal{L}oc_P \arrow[dl, "f"'] \arrow[dr, "g"] & \\
\mathcal{L}oc_M & & \mathcal{L}oc_G
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Bun}_{{}^LP} \arrow[dl, "\varphi"'] \arrow[dr, "\gamma"] & \\
\mathrm{Bun}_{{}^LM} & & \mathrm{Bun}_{{}^LG}
\end{tikzcd}
$$

we get integral transforms

$$\Phi_P = g_! f^* : \text{IndCoh}(\mathcal{L}oc_M, \mathcal{O}) \longrightarrow \text{IndCoh}(\mathcal{L}oc_G, \mathcal{O})$$

$$\text{Eis}_P = \gamma_! \varphi^* : D(\mathrm{Bun}_{{}^LM}, \mathcal{D}) \longrightarrow D(\mathrm{Bun}_{{}^LG}, \mathcal{D})$$

and we should have

$$c_G \circ \text{Eis}_{{}^LP} = \Phi_P \circ c_M$$

<u>Remark</u> - Φ_P, $\text{Eis}_{{}^LP}$ preserve coherence
- $\text{Eis}_{{}^LP}$ preserves perfectness
- Φ_P does not preserve compactness ~~perfectness~~

<u>Guess</u> On the LHS of the geometric Langlands correspondence we want the subcategory of $\text{IndCoh}(\mathcal{L}oc_G, \mathcal{O})$ generated by $\Phi_P(D_{coh}(\mathcal{L}oc_M, \mathcal{O}))$ for all P

How do we understand this subcategory?

<u>Preview</u>: This will be the subcategory consisting of all ind-coh sheaves w/ singular support in the nilpotent cone.

Singular support for Ind Coh sheaves on a derived stack

A derived stack X is defined by

$$h_X : (\mathbb{C}\text{-dga}^{\leq 0}) \longrightarrow \text{simplicial sets}$$

Examples

1) If A is in $(\mathbb{C}\text{-dga}^{\leq 0})$, then ~~take~~ define RSpec A by

$$B \longmapsto \underline{\text{Hom}}(A, B)$$

2) If M is a smooth scheme and $E \to M$ a vector bundle, $s \in H^0(M, E)$, then $X = \text{zeroes}(s)$ is a lci and has a natural derived structure: RX,

$$RX = (M, \mathcal{O}^{\bullet}_{RX})$$

sheaf of dg-algebras s.t

$$H^0(\mathcal{O}_{RX}) = \mathcal{O}_X$$

3) $L^{\bullet}$ is a $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-dg-Lie algebra. Then $L^{\bullet}$ gives a derived stack RX

$$RX := \text{Rspec}(\text{Sym}\, L_{\geq 1}[1], Q) / \exp L^0$$

$$X = \pi_0(RX) = MC(L_1)/\exp L^0$$

Maurer-Cartan elements