The classical limit of geometric Langlands IV (T. Pantev)

(Freiburg, 2 Aug 2012)

Singular support for IndCoh sheaves on a derived stack X

Idea: qcoh sheaves behave like modules over an algebra (filtered)

If X is a nice derived stack, then

$\mathbb{T}_X \in D_{qcoh}(X, \mathcal{O})$ and

$\mathbb{T}_X[-1]$ is a Lie algebra object

The Lie bracket on $\mathbb{T}_X[-1]$ is given by the Atiyah class of X.

Concretely:

symbol map

$$\begin{array}{ccc} \mathbb{T}_X^\vee \otimes \mathcal{H}om(\mathcal{O}_{X^{(1)}}, \mathcal{O}_X)_{\otimes \mathbb{T}_X} & \xrightarrow{\sigma} & \mathrm{End}(\mathbb{T}_X) \overset{L}{\otimes} \mathbb{T}_X \\ \cup & & \cup \\ \mathrm{End}(\mathbb{T}_X) \longrightarrow \mathcal{A}(\mathbb{T}_X) & \longrightarrow & \mathbb{T}_X \rightsquigarrow \end{array}$$

$a(\mathbb{T}_X) \in \mathrm{Ext}^1(\mathbb{T}_X, \mathbb{T}_X^\vee \otimes \mathbb{T}_X)$

$\updownarrow$

So $a(\mathbb{T}_X) : \mathbb{T}_X[-1] \otimes \mathbb{T}_X[-1] \longrightarrow \mathbb{T}_X[-1]$

This gives the Lie bracket.

Also, for every object $F \in D_{qcoh}(X, \mathcal{O})$ we get an Atiyah extension

$a(F) : \mathrm{End}\, F \longrightarrow \mathcal{A}(F) \longrightarrow \mathbb{T}_X \rightsquigarrow$

i.e.,

$$a(F) : \mathbb{T}_{X}[-1] \otimes F \longrightarrow F$$

So $(F, a(F))$ is a module over $(\mathbb{T}_{X}[-1], a(F))$. So $D_{qcoh}(X, U)$ or $\mathrm{IndCoh}(X)$ can be viewed as a category of modules over the filtered algebra $U(\mathbb{T}_X[-1])$

If we can endow F with a good filtration, then $\mathrm{gr}F$ will be a module over $\mathrm{Sym}(\mathbb{T}_X[-1])$ or $\mathrm{gr}F$ will be a sheaf on a derived stack

$$\mathrm{Rspec}(\mathrm{Sym}(\mathbb{T}_{X}[-1])) = \mathrm{Tot}(\mathbb{L}_{X}[1])$$

Assume X is a quasi-smooth derived ~~stack~~ scheme

i.e., $\mathbb{T}_X$ is perfect of amplitude 1.

Example If Z is an algebraic stack which is a lci, then the natural derived enhancement RZ of Z is quasi-smooth.

If X is quasi smooth, then locally $X \simeq RZ$ where Z is a lci scheme.

$Z = \text{zeroes}(s)$

$s \in H^0(M, E)$ ⇝ so RZ just given by the Koszul complex

M: smooth scheme; E: vector bundle

~~tot(Ap^1~~ $\underbrace{\text{tot}(\mathcal{H}^{-1}(\mathbb{L}_{RZ}))}_{= \ \text{sing}(Z)}$ is a subscheme in $\text{tot}(E^\vee)$.

$\text{sing}(Z) \leftarrow$ conical subscheme in $\text{tot}(E^\vee)$

Given $F \in \text{IndCoh}(RZ)$, ~~define~~ we will define

$$\text{ssupp}(F) \subset \text{sing}(Z)$$

as a certain conical subscheme.

<u>Thm</u> (Isik)

$$D_{qcoh}(RZ) \simeq D_{sing}(\text{tot}(E^\vee), w)^{\mathbb{C}^*}$$

$$w : \text{tot}(E^\vee) \longrightarrow \mathbb{C}$$

$$w = p^* s \cdot \lambda$$

$$\rightsquigarrow \text{IndCoh}(RZ) \simeq \widehat{D_{sing}}(\text{tot}(E^\vee)/\mathbb{C}^*, w) = \widehat{D_{sing}}(w^{-1}(0)/\mathbb{C}^*) = \text{IndCoh}(w^{-1}(0)/\mathbb{C}^*) \Big/ D_{qcoh}(w^{-1}(0)/\mathbb{C}^*)$$

<u>Definition</u> $F \in \text{IndCoh}(RZ)$

$$\updownarrow$$

$$F' \in \text{IndCoh}(w^{-1}(0)/\mathbb{C}^*) / D_{qcoh}(w^{-1}(0)/\mathbb{C}^*)$$

$$\text{ssupp} F = \text{supp}(F')$$

4

<u>Remark</u>: $F \in D_{coh}(RZ, \mathcal{O})$, then $\operatorname{ssupp} F \subset \operatorname{Sing}(Z)$
$\operatorname{Sing}(Z) \subset \operatorname{tot}(E^\vee)$

conical in $\operatorname{Sing}(Z)$

and $F \in \operatorname{Perf}(D_{qcoh}(RZ, \mathcal{O}))$
$\iff \operatorname{ssupp} F = 0$.

Given G-reductive. The stack $\mathcal{L}oc$ is an lci stack, so has a natural derived enhancement $R\mathcal{L}oc$.
Explicitly we can describe $R\mathcal{L}oc$ as follows:
fix $x \in C$ and consider $\mathcal{L}oc_{\log x}$ - the moduli stack of (V, ∇), V - principal G-bundle, ∇ meromorphic connection on V w/ logarithmic pole at x.

<u>Fact</u> $\mathcal{L}oc_{\log}$ is a smooth algebraic stack.

<u>Note</u>: $\mathcal{L}oc \subset \mathcal{L}oc_{\log x}$ and is the zero locus of a section of a vector bundle

$(\mathcal{V}, \nabla) \longrightarrow \mathcal{L}oc_{\log x} \times C$

universal local system

∇ - relative connection on $\mathcal{V}$ (differentials in the C direction) and has a 1st order pole at $\mathcal{L}oc_{\log x} \times \{x\} =: D$

$\operatorname{Res}_D(\nabla) \in \Gamma(D, \operatorname{ad} \mathcal{V}|_D)$
on $\mathcal{L}oc_{\log x}$ we have $E = \operatorname{ad}(\mathcal{V}_x)$

$s = \mathrm{Res}_D(\nabla) \in H^0(\mathcal{L}oc_{\log n}, E)$
and $\mathcal{L}oc = \mathrm{zero}(s)$, $R\mathcal{L}oc \simeq (\mathcal{L}oc_{\log n}, (\Lambda^\bullet E^\vee, \sqrt{s}))$

Let $\mathcal{C} \subset \mathrm{tot}(\mathrm{ad}\, \mathcal{Z}^{e}_{x})$ be a conical closed subscheme.

<u>Def</u> $\mathrm{IndCoh}_{\mathcal{C}}(R\mathcal{L}oc, \mathcal{O}) := \{ F \mid \mathrm{ssupp}\, F \subset \mathcal{C} \}$

From the functoriality constraint on c_G: we want the LHS of GLC to be the subcategory in $\mathrm{IndCoh}(R\mathcal{L}oc, \mathcal{O})$ generated by $\Phi_P(\mathrm{Perf}(D_{\mathrm{qcoh}}(R\mathcal{L}oc_M, \mathcal{O})))$ for all parabolics.

<u>Thm</u> (Arinkin-Gaitsgory) This category $= \mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{L}oc, \mathcal{O})$.

nilpotent cone in $\mathrm{ad}\, \mathcal{Z}^{e}_{x}$.

<u>GLC</u>: There exists a functorial equivalence of categories
$c_G : \mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{L}oc, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun}, \mathcal{D})$
intertwining tensorization and Hecke operators.

(<u>Variant</u>: rigidified version)
$c_G : \mathrm{IndCoh}_{\mathcal{N}}(R\underline{\mathcal{L}oc}, \mathcal{O}) \xrightarrow{\sim} D({}^L\mathrm{Bun}, \mathcal{D})$

True for $G = GL_1, GL_2$.

The classical limit conjecture

Assume $G, {}^LG$ are semisimple.

$\mathrm{IndCoh}(R\mathcal{L}oc, \mathcal{O})$ comes in a natural 1-parameter family of categories

$D({}^L\mathrm{Bun}, \mathcal{D})$ "

The first family comes from a 1-parameter deformation of $\mathcal{L}oc$ (or $R\mathcal{L}oc$)

There is a moduli stack $\mathcal{H} \to \mathbb{A}^1$

$\mathcal{H}$ = moduli of (V, ∇, t) where

V - principal G-bundle

$t \in \mathbb{A}^1$

∇ is a flat t-connection on V

$$0 \to \mathrm{ad} V \to \mathcal{A}(V) \underset{\nabla}{\overset{dp}{\rightleftarrows}} T_C \to 0$$

∇ is a t-connection if ∇ is an $\mathcal{O}$-linear map such that

$$dp \circ \nabla = t \cdot \mathrm{id}$$

Dilation by $\mathbb{C}^*$ acts on t-connections

$$z \cdot (V, t, \nabla) = (V, zt, z\nabla).$$

$\rightsquigarrow$ $\mathcal{H} \to \mathbb{A}^1$ $\mathbb{C}^*$-equivariant

Also if (V, t, ∇) is a t-connection, then $(V, \frac{1}{t}\nabla)$ is a connection.

So $\mathcal{H}|_{\mathbb{A}^1 - \{0\}} = \mathbb{A}^1 - \{0\} \times \mathcal{L}oc$

$\mathcal{H}_0$ = stack of 0-connections

$= (V, 0, \nabla)$

$$\nabla : T_C \to \mathcal{A}(V) \supset \mathrm{ad}(V)$$

$$\nabla \in H^0(C, \mathrm{ad}V \otimes \Omega^1 C)$$

$\mathcal{H}_0$ = stack of Higgs bundles.