The classical limit of the geometric Langlands V

(T. Pantev) (Freiburg, 3 Aug 2012)

$G, {}^LG$ - semi-simple

$$\begin{tikzcd}
\mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{L}oc, \mathcal{O}) \arrow[r, "\sim", "c"'] \arrow[d, "\wr"] & D({}^L\mathcal{B}un, \mathcal{D}) \arrow[d, "\wr"] \\
\mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{H}iggs, \mathcal{O}) \arrow[r, "\sim", "c_{\text{classical}}"'] & \mathrm{IndCoh}_{\mathcal{N}}(R{}^L\mathcal{H}iggs, \mathcal{O})
\end{tikzcd}$$

Recall: The moduli stack of t-connections on G-bundles

$$\begin{tikzcd}
\mathcal{H}iggs \arrow[r, hook] \arrow[d] & \mathcal{H} \arrow[d] & (\mathbb{A}^1 - \{0\}) \times \mathcal{L}oc \arrow[l, hook'] \arrow[d] \\
0 \arrow[r, hook] & \mathbb{A}^1 & \mathbb{A}^1 - \{0\} \arrow[l, hook']
\end{tikzcd}$$

$\mathcal{H}iggs$ = moduli of pairs (E, θ)

E - principal G-bundle

$\theta \in H^0(C, \operatorname{ad} E \otimes \Omega^1_C)$, $\underline{\theta \wedge \theta = 0}$

vacuous in case of curves

Aside

$\mathcal{H}iggs = T^\vee \mathcal{B}un$ and $\mathcal{H}iggs$ acts on $\mathcal{L}oc$ by translations

Get a family of derived stacks

$$\begin{tikzcd}
R\mathcal{H} \arrow[d] \\
\mathbb{A}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Sing}(R\mathcal{H}) \supset \mathcal{N} \arrow[d] \\
\mathbb{A}^1
\end{tikzcd}$$

Thus, we get a family of DG-categories

$$\begin{array}{c} \mathrm{IndCoh}_{\mathcal{N}}(R\mathcal{H}/\mathbb{A}^1) \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

and gives the LHS vertical specialization

On the RHS we have a family of categories which is a dequantization.

$\mathcal{D}$ is a filtered sheaf of algebras. so it can be realized as a fibre of a Rees ring.

We have a sheaf of algebras

$$R \longrightarrow {}^L\mathrm{Bun} \times \mathbb{A}^1 \quad \text{s.t.}$$

$$R|_{{}^L\mathrm{Bun} \times (\mathbb{A}^1 - \{0\})} = p_1^* \mathcal{D}$$

$$R|_{{}^L\mathrm{Bun} \times \{0\}} = \mathrm{gr}\, \mathcal{D} = \mathrm{Sym}\, T_{{}^L\mathrm{Bun}}$$

$R \leftrightarrow$ module over $\mathcal{O}_{{}^L\mathrm{Bun}}[t]$

$$R \subset \mathcal{D}[t, t^{-1}]$$

$$R = \left\{ \sum t^{-i} p_i \mid p_i \in \mathcal{D}^{\leq i} \right\}$$

Get a family of categories

$$\begin{array}{c} D({}^L\mathrm{Bun}, R/\mathbb{A}^1) \\ \downarrow \\ \mathbb{A}^1 \end{array}$$

which gives a specialization of $D({}^L\mathrm{Bun}, \mathcal{D})$ to $D({}^L\mathrm{Bun}, \mathrm{Sym}\, T_{{}^L\mathrm{Bun}})$

$$\| $$

$$D(\underbrace{T^\vee {}^L\mathrm{Bun}}_{{}^L\mathrm{Higgs}}, \mathcal{O})$$

To formulate the classical limit conjecture, we need to understand the limit of the tensorization and Hecke operators.

No problem w/ tensorization operators $W^{x,\mu} \rightsquigarrow W^{x,\mu}_{cl}$

$$W^{x,\mu}_{cl}(\mathcal{F}) = \mathcal{F} \otimes \rho^\mu(\mathcal{E}_x)$$

What about the Hecke operators $H^{x,\mu} \rightsquigarrow$?

$$\begin{tikzcd} & {}^L\mathcal{H}ecke^{x,\mu} \arrow[dl] \arrow[dr] & \\ {}^L\mathrm{Bun} & & {}^L\mathrm{Bun} \end{tikzcd}$$

$\leftarrow$ want to interpret these as correspondences acting on $\mathrm{Sym}\, T_{{}^L\mathrm{Bun}}$ - modules

<u>Aside</u> If X a smooth space/stack, then there is an equivalence:

$$sc : \cancel{D} \; D_{qcoh}(T^\vee X, \mathcal{O}) \xrightarrow{\sim} D(\mathrm{Higgs}(X))$$

$$\widetilde{\mathcal{F}} \qquad\qquad (\mathcal{E},\ \theta : \mathcal{E} \to \mathcal{E} \otimes \Omega^1,\ \theta \wedge \theta = 0)$$

$$sc(\mathcal{F}) = p_* \widetilde{\mathcal{F}} \qquad p : T^\vee X \to X$$

<u>Def</u> $IC_{{}^L\mathcal{H}ecke}$ is a mixed Hodge module and has a Hodge filtration which is a good filtration

$$I^{\mu,x}_{cl} = \mathrm{gr}_F\left(IC^{x,\mu}_{{}^L\mathrm{Bun}}\right)$$

This defines

${}^L H^{x,\mu}_{cl}$ = integral transform w/ this kernel

<u>Conjecture</u> $\exists!$ c_{cl} s.t. intertwining $W_{cl}^{x,\mu}$ and ${}^{L}H_{cl}^{x,\mu}$ and is functorial for change of groups

<u>Proof of the classical limit conjecture</u>

Restrict to open substacks where we have no derived structure and no $\mathcal{N}$:

$\mathcal{H}iggs^{reg}$ = stack of regular Higgs Bundle (E, θ) where $\theta_x \in \mathrm{ad}\, E_x \otimes \Omega^1_{C,x}$ is a regular element for all x.

$R\mathcal{H}iggs^{reg} = \mathcal{H}iggs^{reg}$ and $\mathrm{Sing}(\mathcal{H}iggs^{reg}) = 0$

We want to construct

$$c_{cl} : D_{qcoh}(\mathcal{H}iggs^{reg}, \mathcal{O}) \longrightarrow D_{qcoh}({}^{L}\mathcal{H}iggs^{reg}, \mathcal{O})$$

<u>Idea: abelianize both sides</u>

<u>Important fact</u>: $\mathcal{H}iggs^{reg}$ is an abelian group stack. This is Hitchin's abelianization (in the form of Donagi-Gaitsgory). There is a natural morphism $h : \mathcal{H}iggs^{reg} \longrightarrow B$

the Hitchin map

the Hitchin base

smooth variety of dim $= \dim G\,(g-1)$

If (E,θ) is a G-Higgs bundle, $\theta: C \to \mathrm{ad}E \otimes \Omega^1_C$

let
$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[dr, "v(\theta)"'] & \mathrm{ad}E \otimes \Omega^1_C \arrow[d, "v"] \\ & \frac{\mathrm{ad}E}{G} \otimes \Omega^1_C \end{tikzcd}$$

$$\| $$

$$\frac{\mathfrak{g}}{G} \otimes \Omega^1_C = (\mathfrak{t} \otimes \Omega^1_C)/W$$

$\rightsquigarrow$ so θ gives a section
$$\begin{tikzcd} v(\theta) : C \arrow[r] & (\mathfrak{t} \otimes \Omega^1_C)/W \\ \tilde{C} \arrow[u] \arrow[r] & \mathfrak{t} \otimes \Omega^1_C \arrow[u] \end{tikzcd}$$

Get a W-Galois cover $\tilde{C} \to C$ called the cameral cover associated w/ θ.

$B = \Gamma((\mathfrak{t} \otimes \Omega^1_C)/W)$ = moduli of W-cameral covers of C

If $P_1, \dots, P_d$ is a basis of G-invariant polynomials on $\mathfrak{g}$, then $B = \bigoplus_{i=1}^{d} H^0(\cancel{E}\, C, {\Omega^1_C}^{\otimes \deg P_i})$

$h(\theta) := \tilde{C} \leftrightarrow v(\theta) \in \Gamma(C, (\mathfrak{t} \otimes \Omega^1_C)/W)$

given $\tilde{C} \to C$ we get an abelian group scheme on C:

$$\mathcal{T}_{\tilde{C}} = \text{“}(p_*(\mathrm{cochar}(G) \otimes \mathcal{O}^\times))^W\text{”}$$

and if $\tilde{C} = h((E,\theta))$ for (E,θ) regular, then

$\mathcal{T}_{\tilde{C}}$ = centralizer of θ in $\mathrm{Aut}\, E$

Thm $\mathcal{H}iggs^{reg} \simeq B\mathcal{J}$ where $\mathcal{J} \to B \times C$

$\rightsquigarrow$ so $\mathcal{H}iggs^{reg}$ is a commutative group stack over B

There is a natural identification $B = {}^LB$

Hope

$$\begin{tikzcd} \mathcal{H}iggs^{reg} \arrow[dr] & & {}^L\mathcal{H}iggs^{reg} \arrow[dl] \\ & B = {}^LB & \end{tikzcd}$$

are Cartier dual

There are natural symmetries acting on both sides

$W^{x,\mu}_{ab}$ = abelianized tensorization (tensor w/ translati on inv. line bundles)

${}^LH^{x,\mu}_{ab}$ = " Hecke corr. (translation by sections)

$W^{x,\mu}_{ab} = W^{x,\mu}_{cl}$ are the same algebra of endofunctors

Thm (Arinkin - Bezrukavnikov)

$${}^LH^{x,\mu}_{cl} = {}^LH^{x,\mu}_{ab}$$

Thm If $Discr \subset B = {}^LB$ is the discriminant of h (= discriminant of ${}^Lh$), then there is a Poincare ~~sheaf~~ line bundle

$$P \to \mathcal{H}iggs^{reg} \times_{B - Discr} {}^L\mathcal{H}iggs^{reg}$$

which identifies

$$\mathcal{H}iggs^{reg} \simeq ({}^L\mathcal{H}iggs^{reg})^D = Hom({}^L\mathcal{H}iggs^{reg}, BG_m)$$

and gives an equivalence intertwining

$W^{x,\mu}_{ab}$ and ${}^LH^{x,\mu}_{ab}$

Remark fix $\tilde{c}$.

$$h^{-1}(\tilde{c}) = B\mathcal{J}_{\tilde{c}} = \begin{pmatrix}\text{moduli space of} \\ \mathcal{J}_c\text{-bundles on } c\end{pmatrix} \times BZ(G)$$

$$= \underbrace{\pi_0\begin{pmatrix}\text{moduli} \\ \text{space}\end{pmatrix}}_{\pi_1(G)} \times \begin{pmatrix}\text{moduli space} \\ \text{of top. trivial} \\ \mathcal{J}_c\text{-bundles}\end{pmatrix} \times BZ(G)$$

abelian variety $\mathrm{Prym}^G(\mathcal{J}/c)$

$${}^L h^{-1}(\tau) = \pi_1({}^L G) \times \mathrm{Prym}^{{}^L G}(\mathcal{J}/c) \times BZ({}^L G)$$

$$(-)^D : \quad (\pi_1(G))^D = BZ({}^L G) \ ; \ (\mathrm{Prym}^G(\tau/c))^D = \mathrm{Prym}^{{}^L G}(\tau/c)$$

$$(BZ(G))^D = \pi_1({}^L G)$$