

# The classical limit of the geometric Langlands V

(T. Pantev) (Freiburg, 3 Aug 2012)

$G, {}^L G$  - semi-simple

$$\begin{array}{ccc} \text{IndLoc}_{\mathcal{W}}(\mathbb{R}\mathcal{Zoc}, \mathcal{O}) & \xrightarrow[\mathcal{C}]{} & \mathcal{D}({}^L\text{Bun}, \mathcal{D}) \\ \downarrow \} & & \downarrow \} \\ \text{IndLoc}_{\mathcal{W}}(\mathbb{R}\mathcal{Higgs}, \mathcal{O}) & \xrightarrow[\mathcal{C}_{\text{classical}}]{} & \text{IndLoc}_{\mathcal{W}}(\mathbb{R}{}^L\mathcal{Higgs}, \mathcal{O}) \end{array}$$

Recall: The moduli stack of  $t$ -connections on  $G$ -bundles

$$\begin{array}{ccccc} \mathcal{Higgs} & \longleftrightarrow & \mathcal{H} & \longleftarrow & (A' - \{0\}) \times \mathcal{Zoc} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftrightarrow & A' & \longleftarrow & A' - \{0\} \end{array}$$

$\mathcal{Higgs}$  = moduli of pairs  $(E, \theta)$   
 $E$  - principal  $G$ -bundle

$\theta \in H^0(C, \text{ad } E \otimes \Omega_C^1), \quad \theta \wedge \theta = 0$   
 vacuous in case of curves

Aside

$\mathcal{Higgs} = T^V \text{Bun}$  and  $\mathcal{Higgs}$  acts on  $\mathcal{Zoc}$  by translations

Get a family of derived stacks.

$$\begin{array}{ccc} \mathbb{R}\mathcal{H} & \text{sing}(\mathbb{R}\mathcal{H}) \supset \mathcal{W} & \\ \downarrow & \searrow & \\ A' & \rightarrow & A' \end{array}$$

Thus, we get a family of dg-categories

(2)

$\text{IndCoh}_{\mathcal{W}}(\mathbb{R}\mathcal{H}/\mathbb{A}')$  and gives the LHS vertical specialization  
 $\downarrow$   
 $\mathbb{A}'$

On the RHS we have a family of categories which is a dequantization.

$\mathcal{D}$  is a filtered sheaf of algebras. so it can be realized as a fibre of Rees ring.

we have a sheaf of algebras

$$\mathcal{R} \rightarrow \mathcal{L}\text{Bun} \times \mathbb{A}' \quad \text{s.t.}$$

$$\mathcal{R}|_{\mathcal{L}\text{Bun} \times \{0\}} = P_1^* \mathcal{D}$$

$$\mathcal{R}|_{\mathcal{L}\text{Bun} \times \{0\}} = \text{gr} \mathcal{D} = \text{sym } T_{\mathcal{L}\text{Bun}}$$

$$\mathcal{R} \leftrightarrow \text{module over } \mathcal{O}_{\mathcal{L}\text{Bun}}[t]$$

$$\mathcal{R} \subset \mathcal{D}[t, t^{-1}]$$

$$\mathcal{R} = \left\{ \sum t^{-i} p_i \mid p_i \in \mathcal{D}^{\leq i} \right\}$$

Get a family of categories

$$D(\mathcal{L}\text{Bun}, \mathcal{R}/\mathbb{A}')$$

$$\downarrow$$

$$\mathbb{A}'$$

which gives a specialization of  $D(\mathcal{L}\text{Bun}, \mathcal{D})$  to  $D(\mathcal{L}\text{Bun}, \text{sym } T_{\mathcal{L}\text{Bun}})$

||

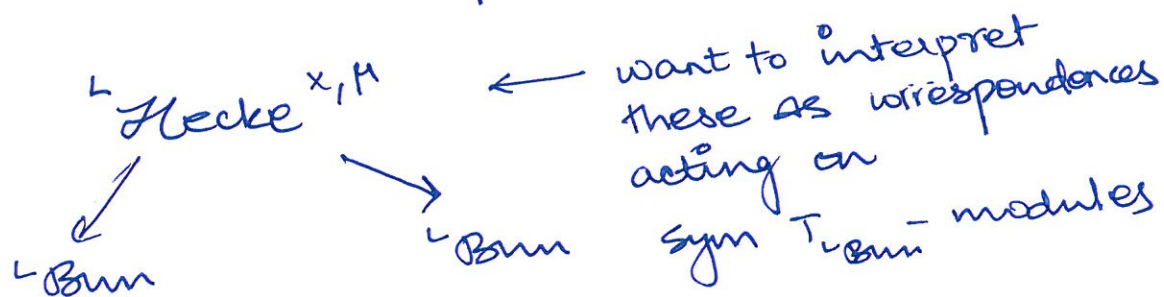
$$D(\underbrace{T^* \mathcal{L}\text{Bun}}_{\text{class}}, \mathcal{O})$$

To formulate the classical limit conjecture, we need to understand the limit of the tensorization and Hecke operators. ③

No problem w/ tensorization operators  $W^{x, M} \rightsquigarrow W_a^{x, M}$

$$W_a^{x, M}(\mathcal{F}) = \mathcal{F} \otimes \rho^M(\mathcal{E}_x)$$

what about the Hecke operators  $H^{x, M} \rightsquigarrow ?$



Aside If  $X$  a smooth space/stack, then there is an equivalence:

$$\text{sc} : \mathcal{D}_{\text{quasi}}(T^*X, \mathcal{O}) \xrightarrow{\sim} \mathcal{D}(\text{Higgs}(X))$$

$\mathcal{F}$

$$(\mathcal{E}, \theta : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^1, \theta \wedge \theta = 0)$$

$$\text{sc}(\mathcal{F}) = p_* \mathcal{F}$$

$$p : T^*X \rightarrow X$$

def  $\mathcal{L} \text{Hecke}$  is a mixed Hodge module and has a Hodge filtration which is a good filtration

$$I_a^{M, X} = \sum_{\mathcal{F}} (\mathcal{I}C_{\mathcal{L} \text{Bun}}^{x, M})$$

This defines

$$\mathcal{L} H_a^{x, M} = \text{integral transform w/ this kernel}$$

④

conjecture  $\exists!$   $c_d$  s.t. intertwining  
 $W_d^{x, \mu}$  and  ${}^L W_d^{x, \mu}$  and  $\circ$  is functorial for change  
of groups

Proof of the classical limit conjecture

Restrict to open substacks where we have no  
derived structure and no  $\mathcal{N}$ :

$\mathcal{Higgs}^{reg} =$  stack of regular Higgs  
bundle  $(E, \theta)$   
where  $\theta_x \in \text{ad } E_x \otimes \Omega_{C, x}^1$   
 $\circ$  is a regular element for  
all  $x$ .

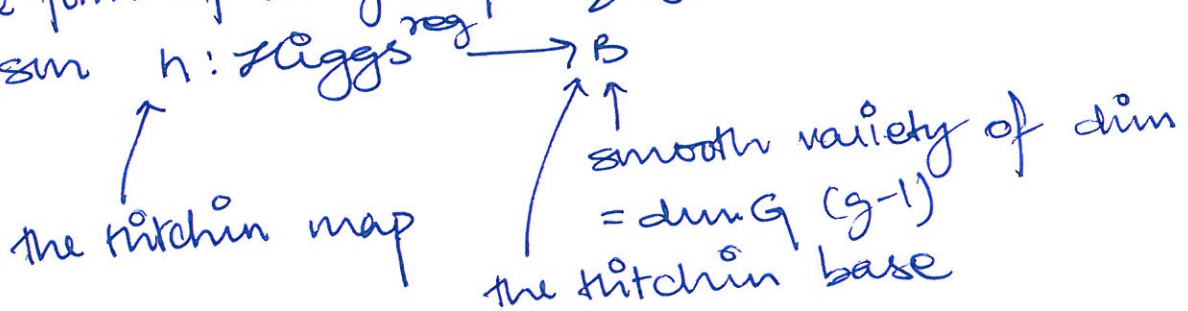
$R\mathcal{Higgs}^{reg} = \mathcal{Higgs}^{reg}$  and  $\text{Sing}(\mathcal{Higgs}^{reg}) = \emptyset$

We want to construct

$c_d: D_{\text{quoh}}(\mathcal{Higgs}^{reg}, \theta) \rightarrow D_{\text{quoh}}({}^L \mathcal{Higgs}^{reg}, \theta)$

Idea: abelianize both sides

Important fact:  $\mathcal{Higgs}^{reg}$  is an abelian  
group stack. This is Hitchin's abelianization  
(in the form of Donagi-Gritsenko). There is a natural  
morphism  $h: \mathcal{Higgs}^{reg} \rightarrow B$



If  $(E, \theta)$  is a  $G$ -Higgs bundle,  $\theta: c \rightarrow \text{ad} E \otimes \Omega_c$

let 
$$\begin{array}{ccc}
 c & \xrightarrow{\theta} & \text{ad} E \otimes \Omega_c \\
 & \searrow v(\theta) & \downarrow v \\
 & & \frac{\text{ad} E}{G} \otimes \Omega_c
 \end{array}$$

$$\frac{\mathfrak{g}}{G} \otimes \Omega_c = (\mathfrak{z} \otimes \Omega_c) / W$$

$\leadsto$  so  $\theta$  gives a section  $v(\theta): c \rightarrow (\mathfrak{z} \otimes \Omega_c) / W$

$\uparrow \qquad \qquad \uparrow$   
 $\tilde{c} \rightarrow \mathfrak{z} \otimes \Omega_c$

Get a  $W$ -Galois cover  $\tilde{c} \rightarrow c$  called the canonical cover associated w/  $\theta$ .

$B = \Gamma((\mathfrak{z} \otimes \Omega_c) / W) =$  moduli of  $W$ -canonical covers of  $c$

If  $P_1, \dots, P_d$  is a basis of  $G$ -invariant polynomials on  $\mathfrak{g}$ , then  $B = \bigoplus_{i=1}^d H^0(c, \Omega_c^{\otimes \deg P_i})$

$h(\theta) := \tilde{c} \leftrightarrow v(\theta) \in \Gamma(c, (\mathfrak{z} \otimes \Omega_c) / W)$   
 given  $\tilde{c} \rightarrow c$  we get an abelian group scheme on  $c: \mathcal{A}$

$$\mathcal{J}_{\tilde{c}} = \left( p_* (\text{cochar}(G) \otimes \theta^x) \right)^W$$

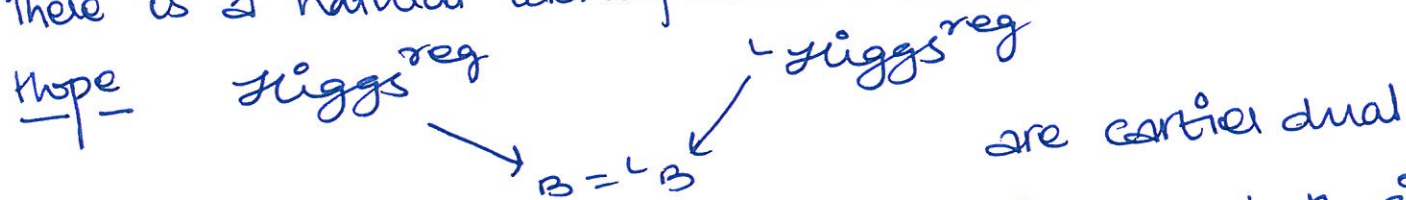
and if  $\tilde{c} = h(E, \theta)$  for  $(E, \theta)$  regular, then

$$\mathcal{J}_{\tilde{c}} = \text{centralizer of } \theta \text{ in } \text{Aut } E$$

Thm  $\text{Higgs}^{\text{reg}} \simeq \text{B.T}$  where  $T \rightarrow B \times C$  (6)

$\leadsto$  so  $\text{Higgs}^{\text{reg}}$  is a commutative group stack over  $B$

There is a natural identification  $B = {}^L B$



There are natural symmetries acting on both sides

$W_{ab}^{X, M} =$  abelianized tensorization (tensor w/ translation on inv. line bundles)

${}^L H_{ab}^{X, M} =$  " Hecke corr. (translation by sections)

$W_{ab}^{X, M} = W_a^{X, M}$  all the same algebra of endofunctors

Thm (Asmkin - Bezrukavnikov)

$${}^L H_a^{X, M} = {}^L H_{ab}^{X, M}$$

Thm If  $\text{Disc} \subset B = {}^L B$  is the discriminant of  $h$  (= discriminant of  ${}^L H$ ), then there is a Poincaré ~~sheaf~~ line bundle

$$P \rightarrow \text{Higgs}^{\text{reg}} \times_{B\text{-Disc}} {}^L \text{Higgs}^{\text{reg}}$$

which identifies  $\text{Higgs}^{\text{reg}} \simeq ({}^L \text{Higgs}^{\text{reg}})^D = \text{Hom}({}^L \text{Higgs}^{\text{reg}}, \mathbb{G}_m)$

and gives an equivalence intertwining

$$W_{ab}^{X, M} \text{ and } {}^L H_{ab}^{X, M}$$

