S. Gaussent   Spherical Hecke algebras for Kac-Moody groups

G - connected, reductive group

$\cup$

K - subgroup

$H(G,K) = \{ f : G/K \longrightarrow \mathbb{Z},\ K\text{-invariant, compactly supported} \}$

For good K this exists. For nice K it is commutative.

(Braverman - Kazhdan) G - affine Kac-Moody group

Bruhat-Tits building

Kapovich - Millson ; Parkinson

joint w/ G. Rosseau

Hovels

G - Kac-Moody group (minimal) over $\mathbb{F}$ (local field)

k = residue field of $\mathbb{F}$

$\cup$

T - maximal torus

$G = G(\mathbb{F})$; $T = T(\mathbb{F})$

$W = W(G,T)$ Weyl group

$Y = \mathrm{Hom}(\mathbb{G}_m, T)$

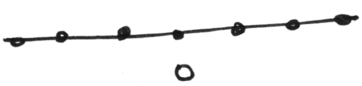
Apartment

$\mathbb{A} = Y \otimes_{\mathbb{Z}} \mathbb{R}$

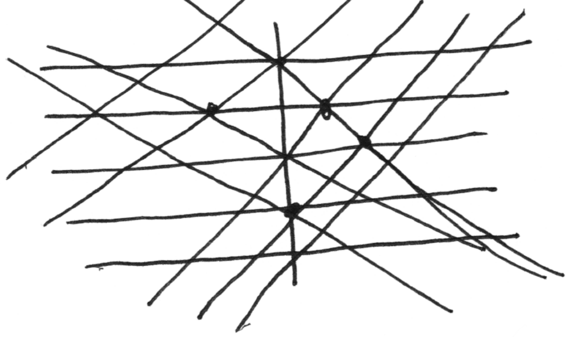
$\forall \alpha \in \underbrace{\Phi^m}_{\text{real roots?}},\ \forall m \in \mathbb{Z}$ $H_{\alpha,m} = \{ x \in \mathbb{A} \mid \alpha(x) + m = 0 \}$

Example

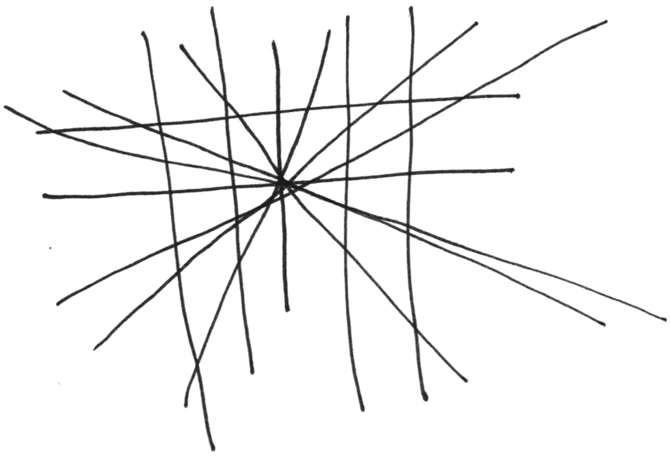
1) Type A_1



2) Type B_2



3) Type $\hat{\tilde{A}}_2$, $G = \hat{SL}_2 \rtimes \mathbb{G}_m$



illustrates problem in Kac-moody setting: dense arrangement

<u>solution</u> filters of subsets of $\mathbb{A}$ (Bruhat-Tits)

$\rightsquigarrow$ simplicial structure.

$C^\vee$ fundamental chamber $\rightsquigarrow \mathcal{T} = \bigcup_{w \in W} w\overline{C^\vee}$ Tits cone

Preorder on $\mathbb{A}$ $x \leq y$ if $y - x$ $\mathcal{T}$

<u>Action of N</u>

$N = N(F) = N_G(T)$

<u>Lemma</u> There is an action of N on $\mathbb{A}$ s.t.:

- T acts by translation
- other elements act by reflections

<u>Parahoric subgroups</u> $\underbrace{\text{(in defining them)}}$

This is partly where the difficulty lies (will ignore)

$\forall\ \Omega$ filters in $\mathbb{A}$ $\rightsquigarrow \hat{P}_\Omega$ parahoric subgroup of G

$\hat{P}_x = \hat{P}_{\{x\}}$ satisfies $\forall x \in \mathbb{A}$

$\hat{P}_x \cap N = \hat{N}_x = \mathrm{Stab}_N(\{x\})$

$m \hat{P}_x m^{-1} = \hat{P}_{m.x}$

<u>Definition</u>

$\mathcal{I} = \mathcal{I}(G, F) = G(F) \times \mathbb{A} / N$ where

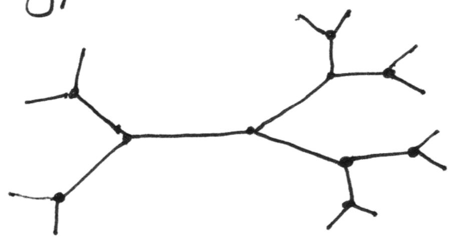
$(g, x) \sim (h, y)$ if $\exists\, m \in N$ st $y = mx$ and $g^{-1}hm \in \hat{P}_x$

consequences: $G(\mathbb{F})$ acts on $\mathcal{J}$ extending the action of N

$\{g\mathbb{A}\}_{g\in G(\mathbb{F})}$ all apartments

$\hat{P}_x = Stab_{G(\mathbb{F})}(\{x\})$

Example 1) Type A_1, $K=\mathbb{F}_2$



any two pts. lie in the same apartment

2) I couldn't draw the picture for type B_2

3) ? speaker says 'impossible to draw'

Properties

1) The preorder extends to a G-invariant preorder on $\mathcal{J}$. If $x\le y$ then x, y are in the same apartment

2) $\rho_{-\infty}: \mathcal{J} \longrightarrow \mathbb{A}$ retraction ($-\infty$ and any x are in the same apartment)

$-\infty$ $\quad -C^N$

$x\le y$ $\quad \rho_{-\infty}([x,y])$ is a Hecke path

Kapovich-Millson

Hecke path = positively folded λ-path ('almost' LS-path)

Spherical Hecke algebra (for a KM-group)

~~K~~ $\kappa=\mathbb{F}_q$ $\quad K=\hat{P}_0 = stab_G(\{0\})$ ($=G(\mathcal{O})$ if G is reductive)

- Weak Cartan decomposition

$G^+ = \{g\in G \mid \underbrace{0 \le g\,0}_{??}\}$ semigroup

$G^+/K = \bigsqcup_{\lambda\in Y^{++}} K\pi^\lambda K/K$

π $\underbrace{\text{uniforms}}_{??}$ K

$Y^+ = Y\cap\mathcal{C}$

$Y^{++} = Y\cap \overline{\mathbf{C}}^N$

vectorial 'distance'

$x \in \mathcal{C}$, x^{++} the unique element in $\overline{C^N}$ s.t.

$\omega \cdot x^{++} = x$

$d^N : \mathcal{I} \times \mathcal{I} \longrightarrow \overline{C^N}$

$(x, y) \longmapsto (g^{-1}y - g^{-1}x)^{++}$

$\mathcal{I}_0^+ = \{\text{vertices of type } 0\} = \{G \cdot 0\} \simeq G^+/K$

$S(0, \lambda) \longleftarrow K\pi^\lambda K/K$

$\{x \in \mathcal{I}_0^+ \mid d(0, x) = \lambda\}$

almost finite support

$f : \mathcal{I}_0^+ \longrightarrow \mathbb{Z}$ has almost finite support if

$\operatorname{supp}(f) \subset \bigcup_{\substack{\text{finite} \\ \lambda}} (\lambda - Q_+^\vee) \cap Y^{++}$, where $Q_+^\vee = \sum_{i \in I} \mathbb{N} \alpha_i^\vee$

Def $\mathcal{H} = \{f : \mathcal{I}_0^+ \longrightarrow \mathbb{Z} \mid f$ K invariant w/ almost finite support$\}$

convolution

$$f * g(z) = \sum_{\substack{0 \le x \le z \\ x \in \mathcal{I}_0^+}} f(x) g(d^N(x, z))$$

$\{c_\lambda = \mathbb{1}_{K\pi^\lambda K/K}\}_{\lambda \in Y^{++}}$ formal basis

$c_\lambda * c_\mu = \sum_\nu m_{\lambda,\mu}^\nu c_\nu$, $m_{\lambda,\mu}^\nu = c_\lambda * c_\mu(\nu)$

$= \#\{\text{triangles } [0, x, \nu] \text{ in } \mathcal{I} \mid 0 \le x \le \nu,\ d^N(0, x) = \lambda,\ d^N(x, \nu) = \mu\}$

Thm Assume $(\alpha_i^\vee)$ are linearly independent. Then

1) $\forall \lambda, \mu, \nu \in Y^{++}$, $m_{\lambda,\mu}^\nu$ is finite
2) $\forall f, g \in \mathcal{H}$, $f * g \in \mathcal{H}$ $\rightsquigarrow$ $\mathcal{H}$ is an algebra
3) θ involution on $G(F)$
4) $\mathcal{H}$ commutative

Braverman-Kazhdan-Patnaik
affine case:

$I \backslash G^+ / I \longleftrightarrow$ DAHA ¿