G. Williamson

## Some examples of parity sheaves

**Parity sheaves** (Juteau, Mautner, W.)

$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$ Whitney stratified complex alg. variety (w/ classical topology)

$i_\lambda : X_\lambda \hookrightarrow X$, $d_\lambda = \dim_{\mathbb{C}} X_\lambda$

$Loc(X_\lambda)$ = local systems of $k$-vector spaces on $X_\lambda$

$D^b_\Lambda(X)$ = $\Lambda$-constructible derived category

A *paraversity* is a function

$$\dagger : \Lambda \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

Fix a paraversity $\dagger$. $F \in D^b_\Lambda(X)$

$F$ is $\dagger$-even if

$\mathcal{H}^m(i_\lambda^? F) = 0$ for all $\lambda$, $m \not\equiv \dagger(\lambda) \bmod 2$

and $? \in \{*, !\}$

Example $\dagger(\Lambda) = 0 \Rightarrow \mathcal{H}^{odd}(i_\lambda^* F) = \mathcal{H}^{odd}(i_\lambda^! F) = 0$

$\Leftrightarrow$ stalks and costalks of $F$ vanish in odd degree.

$F$ is $\dagger$-parity if $F \simeq F_0 \oplus F_1$ w/ $F_0$, $F_1[1]$ $\dagger$-even

Assume

(P) For all $\lambda \in \Lambda$, $\mathcal{L} \in Loc(\mathcal{X}_\lambda X_\lambda)$

$H^{odd}(X_\lambda, \mathcal{L}) = 0$

Thm Under (P), given an indecomposable local system $\mathcal{L} \in Loc(X_\lambda)$ there exists up to isomorphism at most one indecomposable ~~t-parity~~ T-parity $\mathcal{E}^T(\lambda, \mathcal{L})$ extending $\mathcal{L}[d_\lambda]$ on $X_\lambda$ to its closure. Any indecomposable T-parity complex is isomorphic $\mathcal{E}^T(\lambda, \mathcal{L})[m]$ for some $\lambda \in \Lambda$, $\mathcal{L} \in Loc(\mathcal{X}_\lambda X_\lambda)$, $m \in \mathbb{Z}$.

If it exists $\mathcal{E}^T(\lambda, \mathcal{L})$ is a parity sheaf

$$\mathcal{E}^T(\lambda) := \mathcal{E}^T(\lambda, \underline{k}_{X_\lambda})$$

Remarks
1) Motivated by Soergel 'On the relation...'
2) often necessary to consider equivariant complexes and an equivariant version of (P) (eg. nilpotent cones, toric varieties, ...)

# Examples

1) $G$ - connected reductive $/\mathbb{C}$
$\cup$
$B$ - Borel
$\cup$
$T$ - maximal torus

$(W,S)$ - Weyl group and simple reflections

$$X = G/B = \bigsqcup_{w\in W} BwB/B = \bigsqcup_{\lambda\in\Lambda = W}$$

$X_\lambda \simeq \mathbb{C}^{?} \Rightarrow$ (P) is satisfied

have existence thanks to Bott-Samelson resolutions

a) $T = \natural$ $\quad(\natural(\lambda) = 0$ 'constant')

and char $k=0$, then $\mathcal{E}^{\natural}(w) \simeq IC(\bar{X}_w)$

b) $T = \diamond$ $\quad(\diamond(\lambda) = d_\lambda$ 'dimension')

then $\mathcal{E}^{\diamond}(w)$ = indecomposable tilting w/ support $\bar{X}_w$

2) $$X = Gr = G((t))/G[[t]] = \bigsqcup_{\lambda\in \text{dominant weights}} X_\lambda$$

(stratification by $G[[t]]$-orbits

Here $\diamond$-parity = $\natural$-parity

$$\mathrm{Perv}_{G[[t]]}(Gr) \xrightarrow{M.V} \mathrm{Rep}\,{}^{L}G_k$$

$$\begin{array}{ccc} \psi & & \psi \\ \mathcal{E}(\lambda) & \longleftrightarrow & T(\lambda) \end{array}$$

if $p>k+1$

$T(\lambda)$ indecomposable tilting module

The <u>p-canonical</u> <u>basis</u>: From now on only consider $p$-parity sheaves

Fix $X = G/B$ as before (finite)

$\mathcal{H}$ - Hecke algebra

$$\mathcal{H} = \bigoplus_{x \in W} \mathbb{Z}[v^{\pm 1}] H_x$$

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w \\ (v^{-1} - v) H_w + H_{sw} & \text{if } sw < w \end{cases}$$

$\{\underline{H}_w\}$ Kazhdan-Lusztig basis

Given $H = \bigoplus H^i$ a graded f.d. vector space

set $P(H) = \sum \dim H^i v^{-i} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$

For all $x \in W$ set

$${}^p\underline{H}_x = \sum_{y \in W} P(H^\bullet(\mathcal{E}(x)_y)) \, v^{-\ell(y)} H_y$$

$p = \operatorname{char} k$

Eg ${}^0\underline{H}_x = \underline{H}_x$ KL basis

We have

i) $\{{}^p\underline{H}_x\}$ is a basis 'p-canonical basis' (cf. Grojnowski)

ii) ${}^p\underline{H}_x = \sum {}^pm_x^y \underline{H}_y$ w/ ${}^pm_x^y \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ self dual

iii) ${}^p\underline{H}_x \, {}^p\underline{H}_y = \sum {}^p\mu^z_{xy} \, {}^p\underline{H}_z$ w/ ${}^p\mu^z_{xy} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$
self dual

Q: (1) $\boxed{{}^p\underline{H}_x \overset{?}{=} \underline{H}_x}$ $\overset{p>h}{\Longrightarrow}$ Lusztig's conjecture around Steinberg wt. (Soergel)

for affine flag variety $G[[t]]$-orbits $\Rightarrow$ Lusztig's conjecture (Fiebig)
(work in progress)
$\Rightarrow$ James' conjecture (type A)

(2) $\boxed{\text{what is } {}^p\underline{H}_x ?}$ conjecture (Riche-W.)

$\Rightarrow [\Delta_y : IC(\overline{X}_w, \mathbb{F}_p)]$
for ${}^LG/{}^LB$

for affine flag variety $\Rightarrow$ dimensions of all $kS_n$ modules (type A)

(1) for $G[[t]]$-orbits on $Gr$
understood : $\Delta(\lambda) \simeq L(\lambda)$

(2) for $Gr$ $(p>h+1)$
characters of tilting modules for ${}^LG_k$

---

For all $w \in W$ choose a reduced expression
$\underline{w} = s_1 \dots s_{\ell(w)}$ for $w \in W$. Let
$\pi_{\underline{w}} : BS(\underline{w}) \longrightarrow G/B$ be the corresponding Bott-Samelson resolution

Set

$$BS = \bigoplus_{\underline{w} \in W} \pi_{\underline{w}*} \underline{k}_{BS(\underline{w})} [\ell(w)]$$

<u>Thm</u> (Elias-W) $\mathrm{Ext}^\bullet(BS)$ can be described by generators and relations to calculate ${}^p\underline{H}_x$

<u>Thm</u> (Riche-Soergel-W.) $R\mathrm{Hom}^\bullet(BS)$ is formal as a dg-algebra as long as $p > 5/2\, \ell(w_0)$

<u>conjecture</u> this bound is unnecessary.

<u>Some examples</u>

1) type $B_2$ $\overset{s}{\bullet} \Leftarrow \overset{t}{\bullet}$ ${}^p\underline{H}_x = \underline{H}_x$ except

${}^p\underline{H}_{sts} = \underline{H}_{sts} + \underline{H}_s$

2) $\overset{s}{\bullet} \overset{\infty}{\text{———}} \overset{t}{\bullet}$ $\{{}^p\underline{H}_x\} \longleftrightarrow$ characters of tilting modules for $SL_2(\overline{\mathbb{F}}_p)$

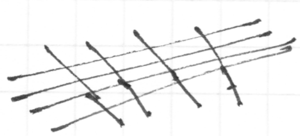
$\tilde{A}_1$

3) $A_n$ : ${}^p\underline{H}_x = \underline{H}_x$ for all $n \leq 6$, all $p$.

(Braden-W.) For $n=7$, ${}^p\underline{H}_x = \underline{H}_x$ for $p \neq 2$

For $p=2$ there are $38/40320$ of $S_8$ w/ ${}^p\underline{H}_x \neq \underline{H}_x$

Five families



$\uparrow w_1$

(Braden, Oberwolfach 2002)

$\uparrow w_4$

Thm (Polo) For all $n$, there exists $w \in S_{4n}$ such that $IC(\overline{X}_w, \mathbb{Z})$ has $n$-torsion in it – a costalk

$\Rightarrow \forall p, \exists w \in S_{4p}$ w/ ${}^p\underline{H}_w \neq \underline{H}_w$ in $S_{4p}$