V. Ginzburg — Equivariant perverse sheaves on the affine grassmanian

joint w/ Simon Riche

$G \supset B \supset T$ $\qquad \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$

$G^\vee \supset B^\vee \supset T^\vee$ Langlands dual

$G^\vee((t)) \supset G^\vee[[t]]$ $\qquad Gr = G^\vee((t))/G^\vee[[t]]$

$G^\vee[[t]]$ orbits $\xleftrightarrow{1-1}$ dominant coweights $X_*(T)$
$\lambda \Downarrow$

$Gr_\lambda = G^\vee[[t]]\cdot\lambda$

Mirković–Vilonen strata $S_\lambda = (U^-)^\vee((t))\cdot\lambda$

$$\begin{tikzcd} \{\lambda\} \arrow[r, hook, "\mathring{i}_\lambda"] \arrow[rr, bend right, "\mathring{i}_\lambda"] & S_\lambda \arrow[r, hook, "\mathring{j}_\lambda"] & Gr \end{tikzcd}$$

Satake category Perv = perverse sheaves $G^\vee[[t]]$-equivariant
$\Downarrow$
$T^\vee$-equivariant

$\mathbb{C}^* \curvearrowright G^\vee[[t]]$ and $\mathbb{C}^* \curvearrowright G^\vee((t))$

So $\mathbb{C}^* \curvearrowright Gr$. Any object of Perv is automatically $\mathbb{C}^*$-equivariant

$A = \mathbb{C}^* \times T^\vee$

$$\tilde{i}_{\lambda !}\,\tilde{i}^{!}_{\lambda}\,\mathring{j}^{!}_{\lambda}P \longrightarrow \mathring{j}^{!}_{\lambda}P \Rightarrow H^*_A(\mathring{i}^{!}_{\lambda}P) \xrightarrow{K_{geom}} H^*_A(S_\lambda, \mathring{j}^{!}_{\lambda}P) = H^*_A(\mathring{j}^{!}_{\lambda}P)$$

Interpret in terms of rep. thy of G

$H^*_A(pt) = \mathbb{C}[\operatorname{Lie} T^\vee][\hbar] = \operatorname{Sym}(\mathfrak{t})\otimes\mathbb{C}[\hbar] =: S_\hbar$

Frac $S_\hbar \otimes K_{geom}$ is an isomorphism.

Bezrukavnikov–Finkelberg: $U(\mathfrak{g}) \supset U(\mathfrak{b}) \supset U(\mathfrak{t})$ have PBW-filtrations. Rees algebras $U_\hbar(\mathfrak{g}) \supset U_\hbar(\mathfrak{b}) \supset U_\hbar(\mathfrak{t}) = S_\hbar$

Harish-Chandra isomorphism $S_\hbar^W \simeq \text{center}(U_\hbar(\mathfrak{g}))$

Universal Verma module $\lambda \in \mathfrak{t}^*$ $\quad$ { There is a left $U_\hbar(\mathfrak{b})$-action on $M_\hbar(\lambda)$

$$M_\hbar(\lambda) = S_\hbar(\lambda) \otimes_{U_\hbar(\mathfrak{b})} U_\hbar(\mathfrak{g})$$

$S_\hbar(\lambda) = S_\hbar$ as a vector space

$\mathfrak{t} \ni t$ acts as mult. $t+\lambda(t)$, $[\mathfrak{b},\mathfrak{b}]$ acts by 0

<u>claim</u> $\exists$ an ad-$\mathfrak{b}$-action on $M_\hbar(\lambda)$ s.t.

$bm - mb = \hbar \, \mathrm{ad}\, b(m) \quad \forall b \in \mathfrak{b}$

ad $\mathfrak{b}$-action is locally finite

$\mathrm{ad}\, t(1\otimes 1) = \lambda(t)\, 1\otimes 1 \quad \forall t \in \mathfrak{t}$

Given a finite dim. $\mathfrak{g}$-rep V get a right $U_\hbar(\mathfrak{g})$ module $V \otimes M_\hbar(\lambda)$.

$U_\hbar(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U_\hbar(\mathfrak{g})$

$\mathfrak{g} \ni x \longmapsto -x\otimes 1 + \hbar \otimes x$

The ~~right~~ left $U_\hbar(\mathfrak{b})$ action on $M_\hbar(\lambda)$ induces an action

$b(v\otimes m) = v \otimes bm.$

$$\begin{tikzcd} (V\otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{b}} \arrow[r, hook] \arrow[rr, bend right, "K_{alg}"'] & (V\otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{t}} \arrow[r, "\mathrm{proj}"] & (V\otimes S_\hbar)^\lambda = V^\lambda \otimes S_\hbar \end{tikzcd}$$

<u>Example</u> $\hbar = 0 \qquad V = \mathbb{C}[G]$

$$\begin{tikzcd} G/T \times \mathfrak{t} \arrow[rr, "a:(gT,t)\mapsto g\times_B t"] \arrow[dr, "q_1"'] & & G\times_B \mathfrak{b} \arrow[dl, "q_2"] \\ & G/B & \end{tikzcd}$$

$\mathcal{O}(\lambda)$ line bundle on G/B

$\Gamma(G\times_B \mathfrak{b}, q_2^*\mathcal{O}(\lambda)) \xrightarrow{a^*} \Gamma(G/T\times \mathfrak{t}, q_1^*\mathcal{O}(\lambda))$.

claim $K_{alg} = a^*$

<u>Geom. Satake equivalence</u> $\quad \mathcal{S} : \mathrm{Rep}\, G \xrightarrow{\sim} \mathrm{Perv}$

Mirkovic-Vilonen implies:

$$\begin{array}{ccc} H_A^\bullet(j_\lambda^! \mathcal{S}(V)) & \overset{MV}{=} & V^\lambda \otimes S_\hbar(\mathfrak{h}t(\lambda)) \\ \uparrow K_{geom} & & \uparrow K_{alg} \\ H_A^\bullet(i_\lambda^! \mathcal{S}(V)) & & (V\otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{b}} \end{array}$$

<u>Thm</u> (G-Riche) Both K_{alg} and K_{geom} are injective and have same image

$$\begin{tikzcd} (V\otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{b}} \arrow[r, "GR", "\sim"'] \arrow[d, "K_{alg}"] & H_A^*(i_\lambda^! \mathcal{S}(V)) \arrow[d, "K_{geom}"] \\ V^\lambda \otimes S_\hbar(\mathfrak{h}t(\lambda)) \arrow[r, "MV", "\sim"'] & H_A^*(j_\lambda^! \mathcal{S}(V)) \end{tikzcd}$$

G/U - base affine space

$G \circlearrowright G/U \circlearrowleft T = B/U$

$\mathcal{D}(G/U) = \bigoplus_{\lambda \in X^*(T)} \mathcal{D}(G/U)^\lambda$ weight decomp. w.r.t right T-action

For any $V \in \mathrm{Rep}\, G$ $\quad [V \otimes \mathcal{D}(G/U)^\lambda]^G = [V \otimes M(\lambda)]^{\mathrm{ad}\,\mathfrak{t}}$

$= \mathrm{Hom}_{U(\mathfrak{g}) \otimes U(\mathfrak{t})}(M(0), V \otimes M(\lambda))$

$\mathcal{D}_\hbar$ = Rees of algebra of $\mathcal{D}(G/U)$

Cor $(V \otimes \mathcal{D}_\hbar^\lambda)^G = (V \otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{t}} = \mathrm{Hom}_{U_\hbar \mathfrak{g} \otimes S_\hbar}(M_\hbar(0), V \otimes M_\hbar(\lambda))$

$(V \otimes M_\hbar(\lambda))^{\mathrm{ad}\,\mathfrak{t}} = H^*_A(i_\lambda^{\circ !} \mathcal{S}(V))$

Gelfand - Graev (1960's). There is a natural action of W = Weyl group on $\mathcal{D}(G/U)$

- by alg. automorphisms
- commutes w/ G-action
- $W \ni w : \mathcal{D}(G/U)^\lambda \longrightarrow \mathcal{D}(G/U)^{w \cdot \lambda}$

used by Kazhdan - Laumon ~ 1985

Same way also have W-action on $\mathcal{D}_\hbar$

Cor $\exists$ W-action on $\mathbb{C}[T^*(G/U)]$

$T^*(G/U) \xrightarrow{\text{moment map}} \mathfrak{g}$

$\cup$

$T^*(G/U)^{\mathrm{reg}}$ = preimage of $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}$

Prop $\exists$ a W-action on $T^*(G/U)^{\mathrm{reg}}$

Proof 1 w/ Simon Riche

Proof 2 w/ David Kazhdan

$$\begin{tikzcd}
(V \otimes \mathcal{D}_\hbar^\lambda)^G \arrow[d, "w"] & = \mathrm{Hom}_{U_\hbar \otimes S_\hbar}(M_\hbar(0), V \otimes M_\hbar(\lambda)) \arrow[d, "w"] \arrow[r, hook, "\mathrm{Kalg}"] & V^\lambda \otimes S_\hbar \\
(V \otimes \mathcal{D}_\hbar^{w\lambda})^G & = \mathrm{Hom}(M_\hbar(0), V \otimes M_\hbar(w\lambda)) \arrow[r, hook, "\mathrm{Kalg}"] & V^{w\lambda} \otimes S_\hbar
\end{tikzcd}$$

cor dynamical Weyl-group action comes from Gelfand-Graev action

conj (Varchenko)