1. Gordon — category $\mathcal{O}$ for rational Cherednik algebras

(joint w/ I. Losev)

- overlap w/ Shan - Vasserot, Stroppel - Webster
- Etingof

1. $W \circlearrowright \mathfrak{h}$

$\uparrow$
complex reflection group

$\mathcal{A} = \{ H = \ker(\alpha_H) \}$, W_H - cyclic

$P = \bigsqcup_{H \in \mathcal{A}/W} \operatorname{Irr}(W_H)' \xleftarrow{\sim} \mathbb{Z}/|W_H|\mathbb{Z} - \{0\}$ $\quad j \mapsto \det^j|_{W_H}$

$\underline{k} \in \mathbb{C}^P$ $\quad H_{\underline{k}}(W)$ subalgebra of $\operatorname{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$

gen'd. by

- W
- multiplication by $\mathbb{C}[\mathfrak{h}]$
- $D_y = \partial_y - \sum_{H \in \mathcal{A}} \alpha_H(y)/\alpha_H \; a_H$ $\quad y \in \mathfrak{h}$

where $a_H = \sum_{p \in \operatorname{Irr}(W_H)'} |W_H| \, k_p \, \varepsilon_p$

$\uparrow$
idempotent in $\mathbb{C}[W_H]$ attached to p

deforms $\mathcal{D}(\mathfrak{h}) \rtimes W$

$\mathcal{O}_{\underline{h}}(W)$ - fin. gend. $H_{\underline{k}}(W)$ - modules, locally nilpotent for D_y $\forall y$

a) highest weight category: $\lambda \in \mathrm{Irr}(W) \rightsquigarrow \Delta_{\underline{k}}(\lambda)$

$$\lambda <_{\underline{k}} \lambda' \text{ iff } \Big(\sum_{H\in\mathcal{A}} a_H\Big)\Big|_{\lambda} - \Big(\sum_{H\in\mathcal{A}} a_H\Big)\Big|_{\lambda'} \in \mathbb{Z}_{>0} \qquad \downarrow L_{\underline{k}}(\lambda)$$

b) $\mathcal{O}_{\underline{k}}(W) \longrightarrow \mathcal{D}(\mathfrak{h}^{reg}) \cancel{\rtimes} W\text{-mod}^{r,\,hol}$

$$\downarrow$$

$$\pi_1(\mathfrak{h}^{reg}/W, *)\text{-mod}$$

$$\parallel$$

$$B_W\text{-mod}$$

$KZ_{\underline{k}} : \mathcal{O}_{\underline{k}}(W) \longrightarrow \mathcal{H}_q(W)$ reps of $\pi_1(\mathfrak{h}^{reg}/W, *)$ satisfying

$$(T_H - 1)\prod_{j=1}^{|W_H|-1}\left(T_H - \exp(2\pi i\, \underline{k}_p - j/|W_H|)\right)$$

fully faithful on projectives

eg $W = S_2 = \langle s \rangle$ $\qquad k = k_1$

$[D_y, x] = 1 - 2ks$

$\Delta(\mathrm{triv}) \longmapsto T \circlearrowleft 1$

$\Delta(\mathrm{sgn}) \longmapsto T \circlearrowleft -\exp(2\pi i k)$

2. $\mathcal{O}_{\underline{k}}(W) \longrightarrow \mathcal{H}_q(W)$ - mod

- 0 - faithful if fully faithful on $\mathcal{O}_{\underline{k}}^{\Delta}$
- 1 - faithful if $\mathrm{Ext}^1_{\mathcal{O}}(M,N) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{H}}(KZ_{\underline{k}}M, KZ_{\underline{k}}N)$

$M,N \in \mathcal{O}_{\underline{k}}^{\Delta}$

<u>Thm</u> (Rouquier, ~~GGOR~~) suppose $\mathcal{H}_q(W_H) \subseteq \mathcal{H}_q(W)$ is semisimple, then $\mathcal{O}_{\underline{k}}(W) \overset{\text{Mor}}{\sim} \mathcal{O}_{\underline{k}'}(W)$ if $\underline{k}' - \underline{k} \in \mathbb{Z}^p$ and $<_{\underline{k}} = <_{\underline{k}'}$

<u>proof</u> 1 - faithfulness $\Rightarrow \mathcal{O}_{\underline{k}}^{\Delta}(W) \xrightarrow{\sim} \mathcal{H}_q(W)^{KZ_{\underline{k}'}\Delta}$

$KZ_{\underline{k}}\Delta = KZ_{\underline{k}'}\Delta$ if $<_{\underline{k}} = <_{\underline{k}'}$

GGOR $\Rightarrow$ 0 - faithful

R $\Rightarrow$ 1 - faithful $\square$

<u>eg</u> $W = S_2$ $\quad \mathcal{O}_{\underline{k}}(W) = \begin{cases} \mathcal{O}_0(\mathfrak{sl}_2) & \underline{k} = 1/2 + \mathbb{Z} \\ \text{s.s} & \text{o.w} \end{cases}$

<u>conjecture</u> $D^b(\mathcal{O}_{\underline{k}}(W)) \xrightarrow{\sim} D^b(\mathcal{O}_{\underline{k}'}(W))$ if $\underline{k} - \underline{k}' \in \mathbb{Z}^p$

3. $D(\mathfrak{h}) \rtimes W \overset{\text{Mor}}{\sim} D(\mathfrak{h})^W$

spherical subalg.

$H_{\underline{k}}(W)$ - mod $\rightleftarrows$ $eH_{\underline{k}}(W)e = U_{\underline{k}}(W)$ - mod

equivalence if $\underline{k} \in$ spherical locus

<u>eg</u> $W = S_2$ $\quad k = \frac{-1}{2}$ $\quad \mathrm{sign} \otimes H_{\underline{k}}(W)$

aspherical

$U_{\underline{k}}(W) = \bigoplus_{i \in \mathbb{Z}} U_k(i)$ $\quad \deg x = 1 \quad \deg \partial y = -1$

eigenspaces for $\underline{eu} = e(\sum x_i \partial y_i)e$

$\mathcal{O}_{\underline{k}}^{sph}$ - fin. gend. $U_{\underline{k}}$-mods, loc. fin. for eu and loc. nilp. for $U(<0)$

$\underline{k}$ - spherical $\quad \mathcal{O}_{\underline{k}}(W) \xrightarrow{\sim} \mathcal{O}_{\underline{k}}^{sph}$

Note: $\operatorname{End}(\mathcal{O}_{\underline{k}}^{sph}) = \operatorname{Rel}\left(U_{\underline{k}}(0) / U_{\underline{k}}(0) \cap UU(<0) \right)$

f. dim alg. flat in $\underline{k}$

4. Geometric symmetries

$U_{\underline{k}}(W)$ is the object quantizing a symplectic singularity

$\Rightarrow$ symmetries of singularities $\rightsquigarrow$ symmetries of $U_{\underline{k}}(W)$

eg. $W = S_2 \qquad U_{\underline{k}}(W) = U(\mathfrak{sl}_2) / \left(\Omega - (k-\frac{1}{2})(k+3/2)\right)$

casimir

$k \longleftrightarrow -k-1$

$$\begin{array}{c} T^*\mathbb{P}^1 \\ \downarrow \\ \mathcal{N} \end{array}$$

$\Rightarrow \mathcal{O}_{\underline{k}}^{sph}(W) \sim \mathcal{O}_{\underline{k}'}^{sph}(W)$ should be expected

$W = G(\ell,1,n) = \mu_\ell^n \rtimes S_n \curvearrowright \mathbb{C}^n \qquad p = (k, \underbrace{k_1, \dots, k_{\ell-1}}_{\circlearrowleft \, S_\ell})$

Thm (G-Losev) $\quad \mathcal{O}_{\underline{k}}(W) \sim \mathcal{O}_{\sigma \underline{k}}(W) \quad \forall \sigma \in S_\ell$

($\underline{k}$ spherical)

s.t. $\Delta_{\underline{k}}(\lambda) \longrightarrow \Delta_{\sigma \cdot \underline{k}}(\lambda^\sigma)$

5. Symmetries from quantizations

Thm (G-Losev) $\quad D^b(\mathcal{O}_{\underline{k}}(W)) \sim D^b(\mathcal{O}_{\underline{k}'}(W))$ if

$\underline{k} - \underline{k}' \in \mathbb{Z}^{\ell}, \; W = G(\ell,1,n)$

Proof $\quad D^b(\mathrm{Coh}\, X) \xrightarrow{\sim} D^b(\mathrm{Coh}^W T^*\mathfrak{h})^{= \mathrm{End}_{\mathcal{O}_X}(P)}$

$$\begin{array}{c} \| \\ \mathbb{C}[T^*\mathfrak{h}] \rtimes W\text{-mod} \end{array}$$

Bezrukavnikov – Kaledin

$$\begin{tikzcd} T^*\mathfrak{h}/W \\ X \arrow[u] \end{tikzcd}$$

$P \rightsquigarrow \hat{P}_\hbar$

$D^b(\mathrm{Coh}\, \mathcal{W}_X^\hbar) \xrightarrow{\sim} D^b(H^\hbar_{\underline{R}}(W))$ equivariant for 2-dim torus

$\Rightarrow$ desired equivalence for $\mathcal{O}$ $\square$