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Hodge theory and real groups

joint w/ W. Schmid

Problem Given a real reductive group $G_{\mathbb{R}}$ determine all the irreducible unitary reps of $G_{\mathbb{R}}$

The unitary dual is known in some cases and the answer is complicated.

Kostant-Kirillov: $\underbrace{\text{one}}^{\text{Idea:}}$ gets all the unitary reps by quantizing coadjoint orbits.

(complexification)

$$\begin{tikzcd} G_{\mathbb{R}} \arrow[r, hook] \arrow[d, leftrightarrow] & G \arrow[d, leftrightarrow] \\ & U_{\mathbb{R}} \arrow[u] \\ K_{\mathbb{R}} \arrow[ur] \arrow[r, hook] & K \end{tikzcd}$$

$X = G/B$ flag variety

$\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$

$$\begin{tikzcd} \lambda \in \mathfrak{h}^* \arrow[r, leftrightarrow] \arrow[d] & \mathfrak{h}^*_{\mathbb{R}} \\ \chi_\lambda \in \mathfrak{Z}[\mathfrak{g}] \arrow[r] & \mathbb{C} \end{tikzcd}$$

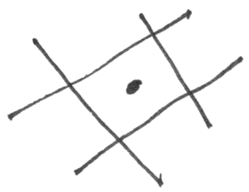
Harish-Chandra

$$\{G_{\mathbb{R}}\text{-representations}\}_{\chi_\lambda} \simeq \{HC\ (\mathfrak{g}, K)\text{-mods}\}$$

An irred. $(\mathfrak{g}, K)$-module is unitarizable iff it posesses a $\mathfrak{g}_{\mathbb{R}}$-invariant +-def Hermitian form.

1st reduction: it suffices to consider real infinitesimal character. We assume this from now on.

In $\mathfrak{h}^*_{\mathbb{R}}$ there is a linear ~~sy~~ subspace $L \subset \mathfrak{h}^*_{\mathbb{R}}$ s.t. for a general $\lambda \in L$ the irred. rep is Hermitian (imagine spherical principal series). Have some linear subspaces on which the rep. reduces and we know that at $\bullet$ the rep. is unitary.



<u>Beilinson - Bernstein</u>

$$\{U(\mathfrak{g})_\lambda\text{-modules}\} \simeq \{\mathcal{D}_{X,\lambda}\text{-modules}\}$$

λ- dominant
λ- irregular

$$\{(\mathfrak{g},K)\text{-modules}\}_{\chi_\lambda} \simeq \{K\text{-equiv } \mathcal{D}_{X,\lambda}\text{-modules}\}$$

$\Rightarrow$ Jantzen conjectures

Here one uses:

a) K-orbits are affinely embedded
b) pointwise purity
c) If $\mathcal{O}$ is a K-orbit, the $\partial\mathcal{O} = \bar{\mathcal{O}} - \mathcal{O}$ is a principal divisor on $\bar{\mathcal{O}}$.

One observes that for λ rational one can reduce the question to a question over $\mathbb{F}_q$ and $\bar{\mathbb{Q}}_\ell$-sheaves
(Maybe should use p-adic Hodge theory?)

Hodge structures and Hodge modules

A complex Hodge structure is a vector space $V/\mathbb{C}$ with a direct sum decomposition

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A polarization is a non-degen. pairing

$$Q : V \times \overline{V} \longrightarrow \mathbb{C} \quad \text{s.t.}$$

a) $V^{p,q}$ are orthogonal wrt Q

b) $(-1)^p (2\pi i)^q Q$ is positive def. on $V^{p,q}$

Remark 1) We do NOT assume that $\dim V < \infty$

2) $\mathbb{C}^{1,0}$ and $\mathbb{C}^{0,1}$ are allowed in the theory and $\mathbb{C}^{1,0} \otimes \mathbb{C}^{0,1} = \mathbb{C}^{1,1}$

A Hodge module is a filtered $\mathcal{D}_X$-module (X - any smooth variety) $(M, F_\bullet M)$, M-regular, holonomic

All the usual functors lift to Hodge-modules

For proper pushforwards filtered $\mathcal{D}$-modules behave in a fixed way but NOT for open ~~set~~ embeddings.

M a Hodge module outside $\{t=0\}$; $j : X - \{t=0\} \hookrightarrow X$

There is a good notion of order of poles for sections of $\mathcal{D}$-modules

It is given by the V-filtration

$t=0$

smooth divisor

$$F_p j_* M = \sum \partial_k^{k} \left(V^{-1} M \cap j_* F_{p-k} M \right)$$

Remark 1) On a pt have fin. dim Hodge structures
2) Variations of Hodge structures are in this category

A polarization is a pairing

$$\varphi : M \otimes \bar{M} \longrightarrow \mathcal{D}b_{X_{\mathbb{R}}} \quad \text{s.t.}$$

$$M \simeq \mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\bar{M}, \mathcal{D}b_{X_{\mathbb{R}}})$$

Remark Because we have filtered duality
$\Rightarrow gr M$ is Cohen-Macaulay on T^*X
$gr(\mathcal{D}) = \mathcal{O}_{T^*X}$
and gr is an exact functor!
Refinement of characteristic cycle!
Refinement to twisted $\mathcal{D}$-modules.

One can extend this to twisted $\mathcal{D}$-modules.
In the case of flag manifolds the usual objects have a canonical lift to mixed Hodge modules as long as λ is REAL

Thm If λ is dominant

$$\Gamma : MHM(X)_\lambda \longrightarrow \{\text{Filtered } U(\mathfrak{g})\text{-modules}\}$$

is exact.

You use Saito's Kodaira Nakano vanishing theorem!

If M is polarized, by integrating

$\varphi : \Gamma(X, M) \otimes \overline{\Gamma(X, M)} \longrightarrow \mathbb{C}$ is $U_{\mathbb{R}}$-invariant!

conj This equips $\Gamma(X, M)$ w/ a polarized Hodge str. (if M irred.)

conj $\Rightarrow$ can produce a Hilbert space w/ $G_{\mathbb{R}}$-action