F. Knop

Orbits on flag varieties

k - algebraically closed field of arbitrary characteristic

G - connected algebraic group$/k$

$\cup$

B - Borel

G/B - flag variety

$K \subseteq G$ - closed subgroup

<u>Problem</u> study K-orbits in G/B

<u>motivation</u>

1) $K = B$ Schubert cells

2) $G_{\mathbb{R}}$ semisimple$/\mathbb{R}$ $K_{\mathbb{R}} \subseteq G_{\mathbb{R}}$ maximal compact

(G, K) = complexification of $(G_{\mathbb{R}}, K_{\mathbb{R}})$

Harish-Chandra modules for $G_{\mathbb{R}} \rightsquigarrow K$-orbits on G/B

K- symmetric $\theta : G \to G$, $\theta^2 = id$, $K = G^{\theta}$

<u>Richardson</u> - Springer char $k \neq 2$

Every Borel subgroup of G contains a θ-stable torus which is unique upto conjugation by $B \cap K$

choose $T_0 \subseteq G$ θ-stable

$$V = \{ g \in G \mid g^{-1}\theta(g) \in N_G(T_0) \}$$

$$V \longrightarrow G/B$$

$$g \longmapsto gB$$

induces a bijection

$$K \backslash V / T_0 \xrightarrow{\sim} K \backslash G / B$$

$$\bar{\varphi} : V \longrightarrow N_G(T_0)$$

$$\varphi : V = K\backslash V / T_0 \longrightarrow N_G(T_0) / T_0 = W$$

$\uparrow$ Weyl group

$$a = g^{-1}\theta(g) \Rightarrow a\theta(a) = 1$$

$$\mathrm{Im}(\varphi) \subseteq \mathcal{J} = \{ \omega \in W / \omega\theta(\omega) = 1 \}$$

$$V \longrightarrow \mathcal{J}$$

<u>example</u> $G = GL(n)$, $K = O(n)$

$T_0 = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$, $\theta(g) = g^{-1}$

$N(T_0) = S_n \cdot T_0 \Rightarrow \theta|_W = id_W$

$\mathcal{J} = \{ a \in S_n \mid a^2 = 1 \}$; $\varphi : K\backslash G/B \xrightarrow{\sim} \mathcal{J}$

<u>Generalize this to arbitrary k</u>

$\mathbb{P}^1$-method.

$B \subsetneq P_1, \ldots, P_r \subseteq G$ minimal parabolics

$\pi_i : G/B \longrightarrow G/P_i$ $\qquad$ fibres $\simeq \mathbb{P}^1$

$$\begin{tikzcd}
K\cdot x \arrow[dr] & & \pi_i^{-1}(\pi_i(K\cdot x)) \subseteq G/B \arrow[dl] \\
& \pi_i(K\cdot x) = K\cdot \pi_i(x) &
\end{tikzcd}$$

$F = \pi_i^{-1}(\pi_i(x)) \simeq \mathbb{P}^1$

K-orbits in $\pi_i^{-1}\pi_i(K\cdot x) \simeq K$-orbits in F

$\simeq S$-orbits in F

$$S = \text{image}\big(K_{\pi_i(x)} \longrightarrow \underset{\substack{\| \\ G_0}}{\mathrm{Aut}(F)} \simeq PGL_2(k)\big)$$

Possibilities for S:

(G) $\quad S = G_0$

(U) $\quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subseteq S \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

(T) $\quad S = \begin{pmatrix} * & \\ & * \end{pmatrix}$

(N) $\quad S = \begin{pmatrix} * & \\ & * \end{pmatrix} \cup \begin{pmatrix} & * \\ * & \end{pmatrix}$

(F) $\quad S$ - finite

Thm (Brion – Vinberg) (Matsuki?)

If $\underbrace{K \text{ has an open orbit}}_{}$ ~~[illegible]~~, then K has only finitely many orbits.

$\uparrow$

K-spherical

For each i one can define an involution on $K\backslash G/B$, denoted $\mathcal{V}$

K-spherical $\Rightarrow$ (F) does not occur

(G)

(U)

(T)

(N)

$s_1, \dots, s_r$ act on $\mathcal{V}$

Thm (K) Let $\operatorname{char} k \neq 2$, $K \subseteq G$ spherical. Then $s_1, \dots, s_r$ defines an action of W on $\mathcal{V}$

(uses W-action on $\overbrace{\text{on}}^{K_0 \text{ of}}$ equivariant constructible sheaves on G/B)

$K \subseteq G$ arbitrary. $Z \subseteq G/B$ closed, K-stable, K-irreducible.

$$m_K(Z) = \text{tr-deg}\ k(Z)^K / k$$

Thm (Vinberg)

$$m_K(Z) \le m_K(G/B)$$

$\mathcal{C}_0 = \{Z \mid m_K(Z) = m_K(G/B)\}$ contains G/B and is finite.

(F) $\cdot$?

Thm ~~char k = 0~~ $\Longrightarrow$ W acts on $\mathcal{C}_0$

char $k \neq 2$

joint w/ G. Pezzini

Reductions: can assume that K is connected, not special, G semisimple of rank 2.

$\mathcal{C}_{00} = \{G/B\}$ in most cases

counterexample in char $k = 2$

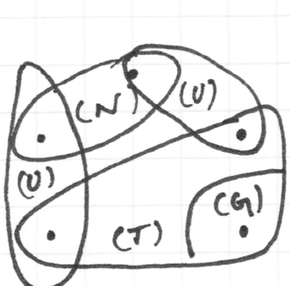
$G = GL(3)$, $K = O(3)$ = stabilizer of $q = x_1x_2 + x_3^2$

$p = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ strange pt. Every line through p is tangent to $\{q = 0\} \subseteq \mathbb{P}^2$

K-orbits in G/B = B-orbits in G/K

= $\{$quadrics in $\mathbb{P}^2\}$
lined

5-orbits



Way out : $\tilde{\mathcal{L}} = \{$K-equivariant double covers of K-orbits in $G/B\}$

<u>Thm</u> char $k \geq 0 \implies W$ acts on $\tilde{\mathcal{L}}$