G - connected, reductive group $\qquad W$ - Weyl group

$\cup$

B - Borel subgroup $\qquad \ell : W \to \mathbb{Z}_{\geq 0}$ (the length function)

$G = \bigsqcup_{w\in W} BwB \qquad G/B = \bigsqcup_{w \in W} BwB/B \qquad G/B \times G/B = \bigsqcup_{w\in W} Z_w$

<u>fake motivation</u>

$$\begin{tikzcd} & Z_w \arrow[dl, "p_w"'] \arrow[dr, "q_w"] & \\ G/B & & G/B \end{tikzcd}$$

$\theta_w := p_{w!} q_w^* : D^b_S(G/B) \to D^b_S(G/B)$

$\uparrow$

$\bigsqcup_{w\in W} BwB/B$ constructible

if $\ell(ww') = \ell(w) + \ell(w')$, then

$\theta_w \circ \theta_{w'} = \theta_{ww'}$.

<u>well known</u> (Beilinson - Bernstein)

$\theta_w : D^b_S(G/B) \xrightarrow{\sim} D^b_S(G/B)$ is an equivalence

<u>Question</u> is $\theta_w : D^b(G/B) \to D^b(G/B)$ an equivalence

<u>convolution</u>

$D^b(B\backslash G/B)$ = $B\times B$-equivariant derived category of G

= B " $\qquad G/B$

= G " $\qquad G/B \times G/B$

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d] \arrow[dr] & G \overset{B}{\times} G \arrow[r, "m"] & G \\ G & G & \end{tikzcd}$$

$M, N \in D^b(B\backslash G/B)$

$M * N = m_!(M \tilde{\boxtimes} N)$

$T_w = i_{w!} \mathbb{C}_{BwB}[\dim_{\mathbb{C}} BwB] \qquad i_w : BwB \hookrightarrow G$

$T_e = \mathbb{1}$ (unit for $*$)

if $\ell(ww') = \ell(w) + \ell(w')$, then $T_w * T_{w'} = T_{ww'}$

<u>Prop</u> $T_w * \mathbb{D}T_{w^{-1}} = \mathbb{1}$

<u>proof sketch</u> 1) $T_w * -$ is left t-exact $\qquad$ essentially because

$- * \mathbb{D}T_w$ " right " $\qquad$ $BwB/B \hookrightarrow G/B$ is affine

2) Easy SL_2 computation:

for $s \in W$ a simple reflection

$T_s * \mathbb{D}T_s = \mathbb{1}$ or something not perverse $\square$

<u>Aside</u> 1) $D^b(B\backslash G/B) \curvearrowright D^b(B\backslash X)$ $\leftarrow$ G-variety

$X = G/B, G/P, \dots$

2) $D^b(B\backslash G/B) = D^b(G\backslash(G/B \times G/B)) \xrightarrow{For} D^b(G/B \times G/B)$

$\rightsquigarrow$ T_w correspond to θ_w

Real motivation

(Ginzburg, Kazhdan-Lusztig)

$$K^{G\times\mathbb{C}^*}(Z) = \text{affine Hecke algebra } \tilde{H}_W$$

$\uparrow$ Steinberg variety

upgrade $D^b(B\backslash G/B)$ to $D^b_m(B\backslash G/B)$

$\uparrow$ mixed Hodge modules

$\rightsquigarrow$ $K_0(D^b(B\backslash G/B))$ = Hecke algebra H_W

$$gr : MHM(G\backslash(G/B\times G/B)) \longrightarrow Coh^{G\times\mathbb{C}^*}(Z)$$

(Tanisaki) $gr : K_0(D^b(G\backslash(G/B\times G/B))) \longrightarrow K^{G\times\mathbb{C}^*}(Z)$

$$H_W \hookrightarrow \tilde{H}_W$$

punchline at the level of categories:

$gr\, T_W$ $\rightsquigarrow$ recover braid group actions constructed by Bezrukavnikov ~~and~~ Riche