

G - connected, reductive group

W - Weyl group

B - Borel subgroup

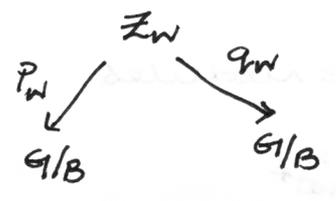
$l: W \rightarrow \mathbb{Z}_{\geq 0}$ (the length function)

$G = \coprod_{w \in W} BwB$

$G/B = \coprod_{w \in W} BwB/B$

$G/B \times G/B = \coprod_{w \in W} \mathbb{Z}_w$

fake motivation



$\Theta_W := P_{W!} q_W^* : D_S^b(G/B) \rightarrow D_S^b(G/B)$

$\coprod_{w \in W} BwB/B$ constructible

if $l(\omega\omega') = l(\omega) + l(\omega')$, then

$\Theta_W \circ \Theta_{W'} = \Theta_{WW'}$

well known

(Beilinson - Bernstein)

$\Theta_W : D_S^b(G/B) \xrightarrow{\sim} D_S^b(G/B)$ is an equivalence

Question

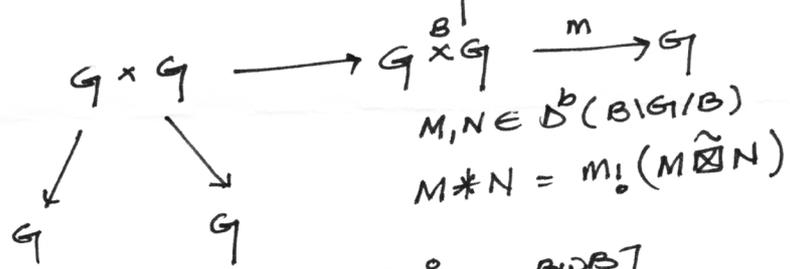
is $\Theta_W : D^b(G/B) \rightarrow D^b(G/B)$ an equivalence

convolution

$D^b(B \backslash G/B) = B \times B$ -equivariant derived category of G

$= B$ " "
 $= G$ " "

G/B
 $G/B \times G/B$



$T_W = i_{W!} \mathbb{1}_{BwB} [\dim_{\mathbb{C}} BwB]$ $i_W : BwB \hookrightarrow G$

$T_e = \mathbb{1}$ (unit for $*$)

if $l(\omega\omega') = l(\omega) + l(\omega')$, then $T_W * T_{W'} = T_{WW'}$

Prop $T_W * \mathbb{1} D T_W^{-1} = \mathbb{1}$

proof sketch

1) $T_W * -$ is left t-exact } essentially because $BwB/B \hookrightarrow G/B$ is affine
 $- * \mathbb{1} D T_W$ " right "

2) Easy SL_2 computation:
 for $s \in W$ a simple reflection

$T_s * \mathbb{1} D T_s^{-1} = \mathbb{1}$ or something not perverse \square

Aside

1) $D^b(B \backslash G/B) \simeq D^b(B \backslash X)$ $\leftarrow G$ -variety

$X = G/B, G/P_1, \dots$

2) $D^b(B \backslash G/B) = D^b(G \backslash (G/B \times G/B)) \xrightarrow{\text{for}} D^b(G/B \times G/B)$

$\rightsquigarrow T_W$ correspond to Θ_W

Real motivation

(Ginzburg, Kazhdan-Lusztig)

$$K^{G \times C^*}(Z) = \text{affine Hecke algebra } \tilde{H}_W$$

↑ Steinberg variety

upgrade

$$D^b(B|G|B) \text{ to } D_m^b(B|G|B)$$

↑ mixed Hodge modules

$$\rightsquigarrow K_0(D^b(B|G|B)) = \text{Hecke algebra } H_W$$

(Tanisaki) $gr: MHM(G|(G/B \times G/B)) \rightarrow \text{coh}^{G \times C^*}(Z)$
 $gr: K_0(D^b(G|(G/B \times G/B))) \rightarrow K^{G \times C^*}(Z)$

$$H_W \hookrightarrow \tilde{H}_W$$

puncture

at the level of categories:

~~gr~~ $gr T_W \rightsquigarrow$ recover braid group actions constructed by Beuzukaw'nikov or Riche