Z. Yun Sheaf theoretic Koszul duality for Kac-Moody groups -1-

(joint w/ R. Bezrukavnikov)

Koszul duality is a symmetry on $D(U\backslash G/B)$

- not visible at level of K_0
- sends IC sheaves to tilting sheaves
- compatible w/ convolution structures

[Beilinson - Ginzburg - Soergel], [Soergel]

Setting:

$D^b_m(X) \leftarrow$ sheaves = { ℓ-adic sheaves on $X/\mathbb{F}_q$; mixed Hodge modules on $X/\mathbb{C}$; sheaves w/ extra filtration $\leadsto$ weights }

$U \subset B \subset G$ - Kac-Moody group (eg. T, SL_2, $\widehat{G((t)) \rtimes \mathbb{C}^*}$, ...)

$U^\vee \subset B^\vee \subset G^\vee$ - Langlands dual

Thm (Bez - Y.) There is an equivalence of dg-categories

$$D^b_m(B\backslash G/B) \xrightarrow{\sim} \hat{D}^b_m(B^\vee \backslash\!\!\backslash G^\vee /\!\!/ B^\vee)$$

s.t.

1) monoidal wrt convolution structures on both sides;

2) it sends $\Delta_w \longmapsto \tilde{\Delta}_w$ (free monodromic standard sheaves)

$\Delta_w = j_{w!}\overline{\mathbb{Q}}_\ell[\ell(w)](\frac{\ell(w)}{2})$

$\nabla_w \longmapsto \tilde{\nabla}_w$

3) it sends $IC_w \longmapsto \tilde{T}_w$ (free monodromic tilting sheaf)

4) $F \longmapsto F(\frac{1}{2})$ lowers weight by 1 (mult. by $q^{-1/2}$ on K_0)

$(\frac{1}{2}) \longmapsto [-1](-\frac{1}{2})$

Notations $G^\vee /\!\!/ B^\vee = (G^\vee/U^\vee)/\!\!/T^\vee$

$D^b(G^\vee /\!\!/ B^\vee) \underset{\text{full}}{\subset} D^b(G^\vee/U^\vee)$

gend. by objects pulled back from $G^\vee/B^\vee$

$T^\vee$-monodromic w/ unipotent monodromy

So, $Sh(T /\!\!/ T)$ = local systems on T w/ unipotent monodromy

so, for $T = \mathbb{C}^*$ $\simeq \mathbb{C}[x]$-mod where x acts nilpotently ($x = (M-1)$)

$\hat{Sh}(T /\!\!/ T) \simeq$ f. gend. $\mathbb{C}[[x]]$-mod w/ modified Homs

(Vilonen comment: ~~unipo~~ version of unipotent Hodge structures)

(Vilonen question: what if you try $\hat{D}^b_m(B \backslash\!\!\backslash G /\!\!/ B) \xrightarrow{\sim} D^b_m(B^\vee\backslash G^\vee/B^\vee)$?

Answer: might work, but price of ^ and $\backslash\!\!\backslash$ completion lose IC sheaves)

Torus example

$D^b(T\backslash T/T) = D^b(pt/T) \overset{\text{Bernstein-Lunts}}{=} \text{dg-mod}/H^*_T(pt)$

$= Sym(\mathfrak{t}^*[-2])$

constant sheaf $\longmapsto Sym(\mathfrak{t}^*[-2])$

$\hat{D}^b(T^\vee \backslash\!\!\backslash T^\vee /\!\!/ T^\vee) \simeq D^b(\text{f. gend } Sym(\mathfrak{t}^\vee)\text{-mod})$

$Sym(\mathfrak{t}^\vee) \longleftrightarrow Sym(\mathfrak{t}^*[-2])$

trade off between hom. degree and weights

after completion:

$\hat{Sh}(T /\!\!/ T) \simeq$ f. gend $\mathbb{C}[[x]]$-mod

$\tilde{\Delta} \longmapsfrom \mathbb{C}[[x]]$

free mono-dromic local system

<u>Tilting sheaves</u>

$U\backslash G/B$ tilting sheaf T is a perverse sheaf on G/B w/ both a Δ-flag and a ∇-flag

<u>Fact</u>: each stratum $w \rightsquigarrow T_w$ supp on $\overline{X_w}$ indecomposable unique up to (non-canonical) isomor-phism

(End(T_w) is large)

IC_w $\qquad$ T_w

$\mathrm{End}(IC_w) = \mathbb{C}$ $\qquad$ $\mathrm{End}_{mix}(T_w) = \mathbb{C}$

$\mathrm{Ext}^\bullet(IC_w, IC_w) \longleftrightarrow \mathrm{End}(T_w)$

<u>Variants of theorem</u>

symmetrizable

$$D^b_m(U\backslash G/B) \xrightarrow{\sim} D^b_m(B^\vee\backslash G^\vee/U^\vee) \xrightarrow{\sim} D^b_m(B\backslash G/U) \overset{g\mapsto g^{-1}}{\underset{\wr}{\downarrow}} D^b_m(U\backslash G/B)$$

$$D^b_m(U\backslash G/B) \xrightarrow{\sim} D^b_m(U\backslash G/B)$$

[BGS]: Koszul-duality sends $IC_w \rightsquigarrow$ projectives $\uparrow$ Radon transform

$IC_w \rightsquigarrow$ tilting

[BY]: na

[BG]: category $\mathcal{O}$ version

[BGS]: parabolic $\mathcal{O} \longleftrightarrow$ singular $\mathcal{O}$ } same for G of finite type

[BY]: $D(U\backslash G/P) \longleftrightarrow D(B^\vee\backslash G^\vee/(U,\psi))$

—— * ——

$G = \widehat{G_0((t))} \rtimes \mathbb{C}^* = \mathbb{G}_m^{rot}$

I = Iwahori (plays role of Borel)

$$\begin{array}{ccc} I & \hookrightarrow & G_0[[t]] \subset G_0((t)) \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & G_0 \end{array}$$

sheaves on total space of det. $\mathbb{G}_m$-torsor on aff. flag

Thm $\Rightarrow$ $D^b_m\left(\underline{I\backslash G_0((t))/I}_{\mathbb{G}_m^{rot}}\right) \xrightarrow{\sim} \overbrace{D^b_m(\hat{I}\backslash \widehat{G_0((t))}/\hat{I})}$

'^' means add central extension

$H^*_{\mathbb{G}_m^{rot}}(pt) \longleftrightarrow$ monodromy along the central torus

$U_G(\det) \longleftrightarrow$ monodromy resulting from $\mathbb{G}_m^{rot}$-action

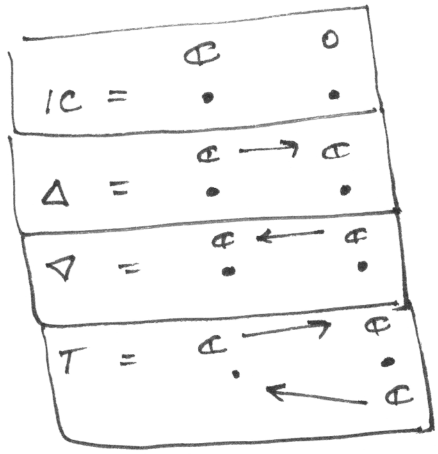
SL_2 case

$\mathrm{Perv}(\mathbb{P}^1; \mathbb{A}^1 \sqcup pt)$

$$= \mathrm{Rep}\left(\begin{tikzcd} \bullet \arrow[r, bend left, "p"] & \bullet \arrow[l, bend left, "q"] \end{tikzcd} ; \; qp = 0 \right)$$

stalk on $\mathbb{A}^1$ — vanishing cycle toward 'pt'.

$$\begin{tikzcd} & \Delta \arrow[dl] \arrow[dr, hook] & \\ IC = \mathrm{const} \arrow[dr, hook] & & T \arrow[dl] \\ & \nabla & \end{tikzcd}$$



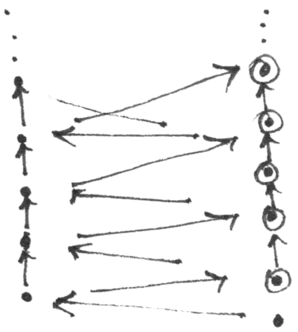
$D(B\backslash G/B)$ for SL_2

$$\begin{tikzcd} \bullet \arrow[loop left, "x"] \arrow[r, bend left, "p"] & \bullet \arrow[l, bend left, "q"] \arrow[loop right, "x"] \end{tikzcd} ; \quad qp = x$$

p, q commute w/ x

Soergel question: meaning of 'm' in D_m. Mixed = mixed Tate?

Answer: all mixed sheaves allowed, not just Frobenius semi-simple



$\tilde{T}_s$ - free monodromic tilting