# Structure of highest weight representations

$\mathfrak{g}$ - f.d simple Lie algebra / $\mathbb{C}$

$\cup$

$\mathfrak{b}$ - Borel subalgebra

$\cup$

$\mathfrak{h}$ - Cartan subalgebra

$\mathfrak{n}^+ = [\mathfrak{b}, \mathfrak{b}]$

$R$ - set of roots (= non-zero eigenvalues of $\mathfrak{h} \curvearrowright \mathfrak{g}$)

$R^+$ - positive roots (= non zero eigenvalues of $\mathfrak{h} \curvearrowright \mathfrak{n}^+$)

in particular,

$$\mathfrak{g} = \underbrace{\bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}}_{\mathfrak{n}^-} \oplus \underbrace{\mathfrak{h} \underbrace{\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}_{\mathfrak{n}^+}}_{\mathfrak{b}}$$

$$\mathfrak{g}_\alpha = \{ g \in \mathfrak{g} \mid [h, g] = \alpha(h) g, \text{ for all } h \in \mathfrak{h} \}$$

$W$ - Weyl group.

<u>example</u> $\mathfrak{g} = sl_2 = \left\{ \begin{pmatrix} t_1 & * \\ * & t_2 \end{pmatrix} \middle| \Sigma t_i = 0 \right\}$

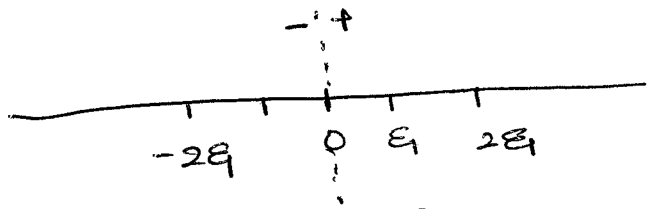
$$\mathfrak{b} = \left\{ \begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \middle| \Sigma t_i = 0 \right\}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \middle| \Sigma t_i = 0 \right\}$$

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \right\}$$

$R = \{ \pm 2\varepsilon \}$ where $\langle \varepsilon, \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \rangle = t_1$

$R^+ = \{2\varepsilon_1\}$ ; $W = S_2$




<u>example</u> $\mathfrak{g} = \mathfrak{sl}_3 = \left\{ \begin{pmatrix} t_1 & & * \\ & t_2 & \\ * & & t_3 \end{pmatrix} \middle| \sum t_i = 0 \right\}$

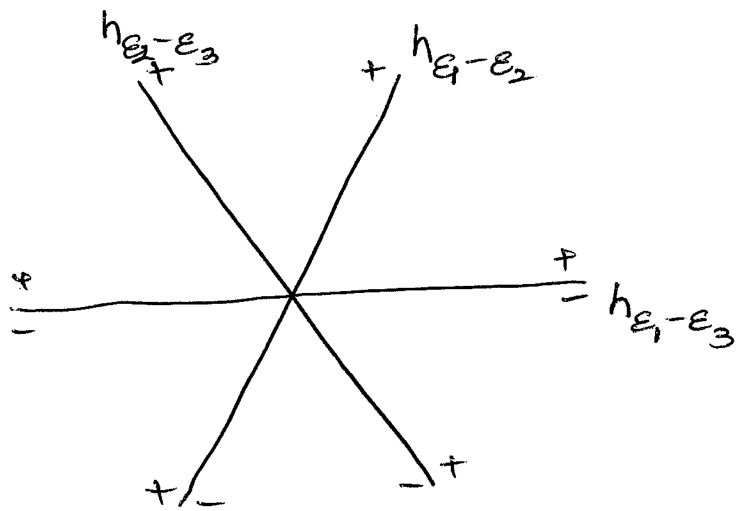
$\mathfrak{b} = \left\{ \begin{pmatrix} t_1 & & * \\ & t_2 & \\ 0 & & t_3 \end{pmatrix} \middle| \sum t_i = 0 \right\}$

$\mathfrak{h} = \left\{ \begin{pmatrix} t_1 & & 0 \\ & t_2 & \\ 0 & & t_3 \end{pmatrix} \middle| \sum t_i = 0 \right\}$

$R = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq 3,\ i \neq j \}$

$R^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 3 \}$

$W = S_3$



<u>Representation</u> <u>Theory</u>

$U(\mathfrak{g})$ - algebra (w/1, associative) generated by $\mathfrak{g}$ and rels

$$xy - yx = [x,y] \quad \text{for all } x,y \in \mathfrak{g}$$

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$$

A <u>$\mathfrak{g}$-module</u> is a $U(\mathfrak{g})$-module.

~~<u>example</u> let $V$ be the $\mathfrak{g} = sl_2$~~

$$\cancel{V = \mathbb{C}\text{-span}\{v_1, v_{-1}\}}$$

$$\cancel{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} v \quad \text{set } e = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \; f = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}}$$

$$\cancel{b = h}$$

<u>example</u> $U(sl_2)$ has generators $e, f, h$ and rels

$$ef - fe = h, \quad he - eh = 2e, \quad hf - fh = -2f$$

Let $V = \mathbb{C}\text{-span}\{v_1, v_{-1}\}$

$$ev_1 = fv_{-1} = 0 \qquad \cancel{ev_{-1}} = ev_{-1} = v_1$$

$$fv_{+1} = v_{-1} \qquad hv_i = -i v_i$$

$$V = \mathbb{C}[x,y]$$

$$e \mapsto x\frac{\partial}{\partial y} \qquad f \mapsto y\frac{\partial}{\partial x}$$

~~given~~ gives a $sl_2$-action $V$

Note: $h = ef - fe$ acts on $x^m y^n$ by the scalar $m - n$

$$V = \bigoplus_{\lambda \in \mathbb{Z}_{\geq 0}} L(\lambda), \text{ where}$$

$$L(\lambda) = \{ p \in \mathbb{C}[x,y] \mid \deg(p) = \lambda \}$$

* Each $L(\lambda)$ is simple
* If $L$ is any f.d simple $sl_2$-module, then $L \simeq L(\lambda)$ for some $\lambda$

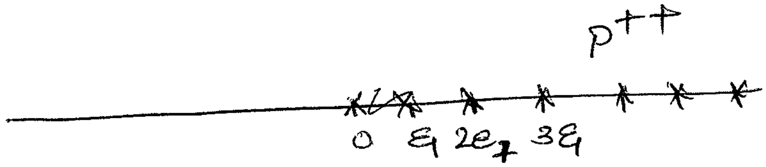
Define
$$P^{++} = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha^\vee, \alpha \in R^+ \}$$

where $\langle \lambda, \alpha^\vee \rangle$ = distance of $\lambda$ from the $\alpha$-hyperplane

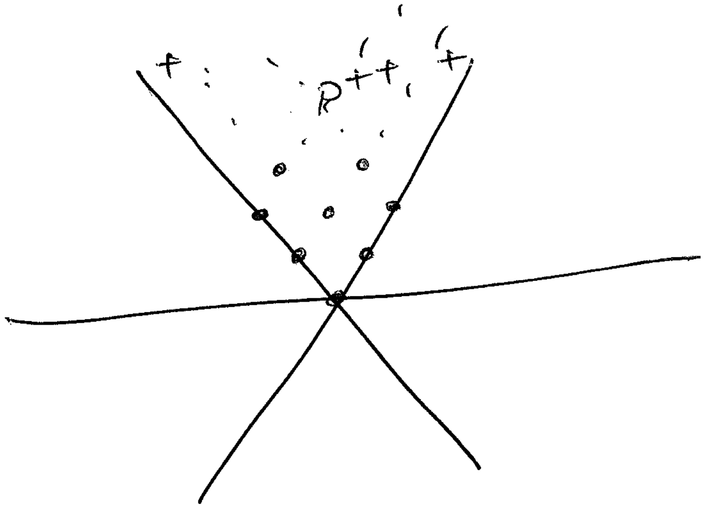
$$\left\{ \text{of } \begin{Bmatrix} \text{simple, f.d} \\ \mathfrak{g}\text{-modules} \end{Bmatrix} \right\} \xleftrightarrow{1-1} P^{++}.$$

$$L(\lambda) \longleftrightarrow \lambda$$

$\underline{sl_2}$




$\underline{sl_3}$




weight $= \mathfrak{h}$-eigenvalue

$\underline{\text{prop}}$ If $V$ is a f.d $\mathfrak{g}$-module, then

1) $V$ is completely reducible
2) $V$ is the direct sum of its weight-spaces
3) If $\lambda$ is a ~~wt~~ weight of $V$ then so is $\omega(\lambda)$ for all $\omega \in W$.

category $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-modules $M$ s.t.

1) $M$ is finitely generated
2) $M$ is the direct sum of its wt. spaces
3) $M$ is $\mathfrak{n}^+$ locally nilpotent

A vector $v_\lambda^+ \in M$ such that $\mathfrak{n}^+ v_\lambda^+ = 0$ ~~is~~ ~~and~~ $\cancel{h v_\lambda^+ = \lambda(h)}$ is a highest weight vector.

* All f.d. simple modules are in $\mathcal{O}$

Let $\lambda \in \mathfrak{h}^*$, then the verma module w/ highest wt. $\lambda$ is

$$\Delta(\lambda) = U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module w/ action given by $\mathfrak{b} \longrightarrow \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$.

* $\Delta(\lambda) \in \mathcal{O}$

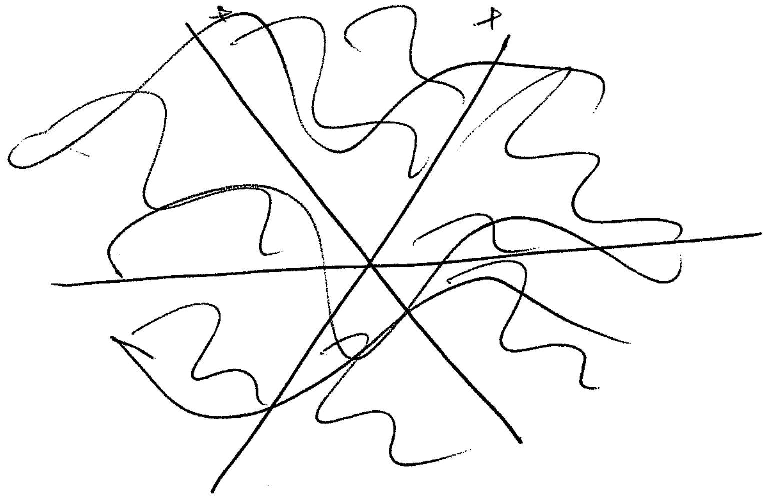
* $\Delta(\lambda)$ is a free $U(\mathfrak{n}^-)$ module $(\Delta(\lambda) = U(\mathfrak{n}^-) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda$

* For any $\mathfrak{g}$-module $M$,

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), M) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, M)$$

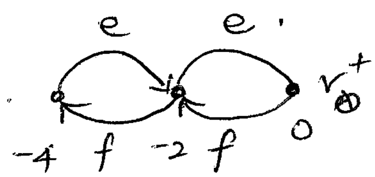
* $\Delta(\lambda)$ has a unique simple quotient, denoted $L(\lambda)$.

$sl_3$



<u>example</u> $sl_2$

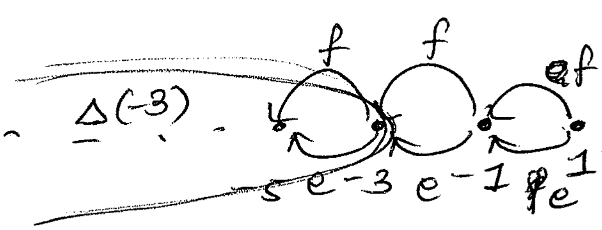
$\Delta(0)$




$$e(f v_0^+) = (fe + [e,f])\, v_\lambda^+$$
$$= 0.$$

$\Delta(-2) \in \Delta(0)$ , $\Delta(0)/\Delta(-2) \simeq L(0)$.

$\Delta(1)$




$\Delta(-3) \subset \Delta(1)$ ; $\Delta(1)/\Delta(-3) \simeq L(1)$

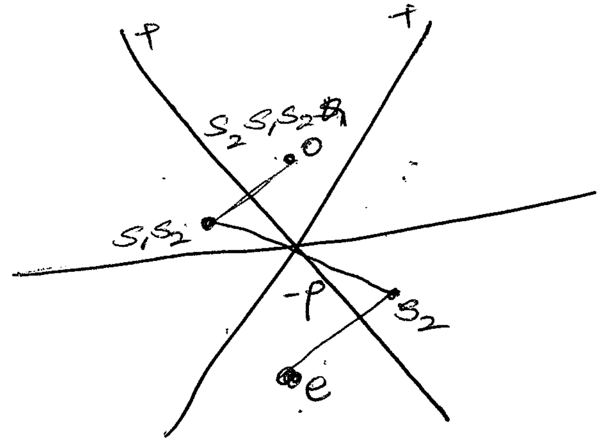
Let $\rho = \frac{1}{2}\sum_{\alpha\in R^+} \alpha$.

~~Set~~ $\cancel{\rho^+ = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z},\ \langle\lambda+\rho,\alpha^\vee\rangle \in \mathbb{Z}_{\geq 0}\}}$

<u>Prop</u> Let $\lambda \in \mathfrak{h}^*$. ~~suppose~~ If $\cancel{\langle\lambda,\alpha^\vee\rangle\in}$ $\langle\lambda+\rho,\alpha^\vee\rangle \in \mathbb{Z}_{\geq 0}$, then $\Delta(s_\alpha\cdot\lambda) \subset \Delta(\lambda)$,

where
$$s_\alpha\cdot\lambda = s_\alpha(\lambda+\rho) - \rho$$

<u>$\mathfrak{sl}_3$</u>




$$\Delta(e) \subset \Delta(s_2) \subset \Delta(s_1 s_2) \subset \Delta(s_2 s_1 s_2)$$

A <u>highest wt. module</u> is a quotient of a verma module.

Every object of $\mathcal{O}$ has a finite filtration w/ factors isomorphic to highest wt. modules

## Harish-Chandra isomorphism.

~~Let~~ $Z$ - center of $U(\mathfrak{g})$

Let $z \in Z$, then $z$ defines an element in $\mathrm{End}_{\mathfrak{g}}(\Delta(\lambda)) = \mathbb{C}$

~~Define~~ Let $S(\mathfrak{h}) = \underbrace{\text{regular } \cancel{\text{pol}}}$ fns. on $\mathfrak{h}^*$ and define

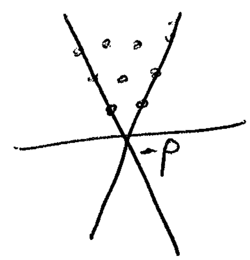
$$\chi : Z \longrightarrow S(\mathfrak{h})$$

$$z \longmapsto \chi_z$$

where $\chi_z(\lambda)\, \Delta(\lambda) = z\, \Delta(\lambda)$.

~~Now for λ~~

set ~~Let~~ $P^+ = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda+\rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$, for all $\alpha \in R^+ \}$.




Let $\lambda \in P^+$, then

$$\Delta(w \cdot \lambda) \subset \Delta(\lambda)$$

So $\chi_z(\lambda) = \chi_z(w \cdot \lambda)$.

Thus

$$\chi : Z \longrightarrow S(\mathfrak{h})^{W\cdot}$$

<u>Thm</u> (Harish-Chandra) The map

$$\chi : Z \longrightarrow S(\mathfrak{h})^{W\cdot}$$

is an isomorphism.

**Cor** Every object of $\mathcal{O}$ has finite length

**Cor** $\mathcal{O} = \bigoplus_{\lambda \in P^+} \mathcal{O}_\lambda$, where

$\mathcal{O}_\lambda$ ~~consists~~ is the serre subcategory of $\mathcal{O}$ generate by $L(w \cdot \lambda)$, $w \in W$.

**Cor** Let $\lambda \in P^+$, then $\Delta(\lambda)$ is projective

**Cor** Let $\lambda \in P^+$, then $\Delta(w_0 \cdot \lambda) \cong L(w_0 \cdot \lambda)$. where $w_0 \in W$ is the longest element of $W$.