ADJOINT FUNCTORS AND TRIANGULATED FUNCTOR CATEGORIES

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1. Reminders and notation

When dealing with categories we will not worry about set theoretical issues. We assume that we remain in a given universe or, as put in [GeMa, p. 38], "that all the required hygiene regulations are obeyed".

1.1. **Notions concerning functors.** Let $F, G : \mathcal{A} \to \mathcal{B}$ be functors between categories $\mathcal{A}$ and $\mathcal{B}$. A morphism of functors $\varphi : F \to G$ consists of a morphism $\varphi_X : F(X) \to G(X)$ for each $X \in \mathcal{A}$, such that for every morphism $f : X \to Y$ the diagram below commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\varphi_X"'] & F(Y) \arrow[d, "\varphi_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

We use the terms 'functorial', 'natural' and 'canonical' as synonyms for 'a morphism of functors' (with the functors in question being obvious from the context). The identity endomorphism, of a functor F, will be denoted $\mathbb{1}_F$.

1.2. **Additive categories and complexes.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. We write $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ for the category of additive functors from $\mathcal{A}$ to $\mathcal{B}$. Functors between additive categories will always be assumed to be additive.

Let $\mathcal{A}$ be an additive category. We use cohomological notation for complexes. That is, a complex $X^\bullet$ in $\mathcal{A}$ is a sequence of morphisms

$$\cdots \longrightarrow X^i \xrightarrow{d_i} X^{i+1} \longrightarrow \cdots ,$$

such that $d_{i+1} \circ d_i = 0$ for each i. The morphisms d_i, collectively, are referred to as the *differential* of $X^\bullet$. If $\mathcal{A}$ is an abelian category, the cohomology $H^*(X^\bullet)$ of $X^\bullet$ is the sequence of objects

$$H^i(X^\bullet) = \frac{\ker(d_i)}{\mathrm{im}(d_{i-1})}.$$

Denote by $\mathrm{Comp}^*(\mathcal{A})$, $* = \emptyset, -, +, b$, the category of all complexes, bounded above complexes, bounded below complexes and bounded complexes in $\mathcal{A}$, respectively. A *chain map* is a morphism in $\mathrm{Comp}(\mathcal{A})$. Viewing each object as a complex concentrated in degree 0, we regard $\mathcal{A}$ as a subcategory of $\mathrm{Comp}(\mathcal{A})$. This is a full and faithful embedding, i.e., $\mathrm{Hom}_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{\mathrm{Comp}(\mathcal{A})}(X, Y)$, for each $X, Y \in \mathcal{A}$.

The shift functor $?[1] : \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{A})$ is defined as follows: if $X^\bullet$ is a complex with differential d, then $(X^\bullet[1])^i = X^{i+1}$ with differential $d'_i = -d_{i+1}$. It is clear that $?[1]$ is a self-equivalence of $\mathrm{Comp}(\mathcal{A})$. For $n \in \mathbb{Z}$, set $[n] = [1]^n$.

1.3. **Double complexes.** Let $\mathcal{A}$ be an additive category. A *double complex* is an object of $\mathrm{Comp}(\mathrm{Comp}(\mathcal{A}))$. Explicitly, a double complex $X^{\bullet,\bullet}$ is the data of $\{X^{i,j}, d'_{i,j}, d''_{i,j}\}_{i,j\in\mathbb{Z}}$ where $X^{i,j} \in \mathcal{A}$ and the pair of 'differentials' $d'_{i,j} : X^{i,j} \to X^{i,j+1}$, $d''_{i,j} : X^{i,j} \to X^{i+1,j}$ satisfy:

$$d''_{i,j+1} \circ d'_{i,j} = 0, \quad d'_{i+1,j} \circ d''_{i,j} = 0, \quad d''_{i,j+1} \circ d'_{i,j} = d'_{i+1,j} \circ d''_{i,j}.$$

$$\begin{tikzcd}
 & & & \\
\cdots \arrow[r] & X^{i,j+1} \arrow[u] \arrow[r, "d''_{i,j+1}"] & X^{i+1,j+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & X^{i,j} \arrow[u, "d'_{i,j}"] \arrow[r, "d''_{i,j}"'] & X^{i+1,j} \arrow[u, "d'_{i+1,j}"'] \arrow[r] & \cdots \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

Given a double complex $X^{\bullet,\bullet}$, we say that $X^{i,j}$ is in *bidegree* (i, j).

Define $\mathrm{Tot} : \mathrm{Comp}(\mathrm{Comp}(\mathcal{A})) \to \mathrm{Comp}(\mathcal{A})$ by

$$\mathrm{Tot}(X^{\bullet,\bullet})^k = \bigoplus_{i+j=k} X^{i,j}, \quad \text{with differential given by} \quad \begin{pmatrix} d'_{i,j} \\ (-1)^i d''_{i,j} \end{pmatrix} : X^{i,j} \to X^{i,j+1} \oplus X^{i+1,j}.$$

The complex $\mathrm{Tot}(X^{\bullet,\bullet})$ is the *total complex* associated to $X^{\bullet,\bullet}$. The object $\mathrm{Tot}(X^{\bullet,\bullet})$ is well defined as long as, for each $k \in \mathbb{Z}$, there are only finitely many $i, j \in \mathbb{Z}$ with $i + j = k$, such that $X^{i,j} \neq 0$, or alternatively, if $\mathcal{A}$ admits countable direct sums.

Let $X^\bullet, Y^\bullet$ be complexes in $\mathcal{A}$ with differentials d'_i and d''_i, respectively. Let $\phi : X^\bullet \to Y^\bullet$ be a chain map. Consider the following double complex:

$$\begin{tikzcd}
 & 0 & 0 & \\
\cdots \arrow[r] & Y^i \arrow[u] \arrow[r, "d''_i"] & Y^{i+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & X^i \arrow[u, "\phi_i"] \arrow[r, "d'_i"'] & X^{i+1} \arrow[u, "\phi_{i+1}"'] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where X^i is in bidegree $(i, -1)$. The *mapping cone* or, simply, the *cone* of ϕ is the total complex of this double complex. We denote it by $\mathrm{cone}(\phi)$. Explicitly,

$$\mathrm{cone}(\phi)^i = Y^i \oplus X^{i+1} \quad \text{with differential} \quad d_i = \begin{pmatrix} d''_i & \phi_{i+1} \\ 0 & -d'_{i+1} \end{pmatrix}.$$

Define

$$\iota : Y^\bullet \to \mathrm{cone}(\phi) \quad \text{by} \quad \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix} : Y^i \to Y^i \oplus X^{i+1},$$

and

$$\delta : \mathrm{cone}(\phi) \to X^\bullet[1] \quad \text{by} \quad \begin{pmatrix} 0 & \mathrm{id} \end{pmatrix} : Y^i \oplus X^{i+1} \to X^{i+1}.$$

Both ι and δ are chain maps. A *standard triangle* is a sequence of morphisms of the form

$$X^\bullet \xrightarrow{\phi} Y^\bullet \xrightarrow{\iota} \mathrm{cone}(\phi) \xrightarrow{\delta} X^\bullet[1]. \tag{1.1}$$

1.4. **Triangulated categories.** A triangulated category consists of an additive category $\mathcal{T}$, endowed with the following structure:

- *shift functor*: a fixed equivalence $?[1] : \mathcal{T} \to \mathcal{T}$;
- *distinguished triangles*: a class of morphisms of the form $X \to Y \to Z \to X[1]$.

This structure is required to satisfy certain additional axioms. For these and the basic properties of triangulated categories we refer the reader to [KaSc, Ch. 10]. We often write $X \to Y \to Z \rightsquigarrow$ to emphasize that a sequence of morphisms is a distinguished triangle. For $n \in \mathbb{Z}$, set $[n] = [1]^n$ and for all $X, Y \in \mathcal{T}$, put

$$\mathrm{Ext}^n(X,Y) = \mathrm{Hom}_{\mathcal{T}}(X, Y[n]).$$

Let $\mathcal{T}'$ be another triangulated category with shift functor $[1]'$. A functor $F : \mathcal{T} \to \mathcal{T}'$ is *exact* if F preserves distinguished triangles and such that there exists a canonical isomorphism $F \circ [1] \simeq [1]' \circ F$. The category of exact functors from $\mathcal{T}$ to $\mathcal{T}'$ is denoted $\mathcal{H}\mathrm{om}_{\mathfrak{Tr}}(\mathcal{T}, \mathcal{T}')$.

The *Grothendieck group* $K_0(\mathcal{T})$, of a triangulated category $\mathcal{T}$, is the free abelian group on symbols $[X], X \in \mathcal{T}$, modulo the relation $[X] = [X_1] + [X_2]$ for each distinguished triangle $X_1 \to X \to X_2 \rightsquigarrow$. The axioms of a triangulated category imply that $[X[1]] = -[X]$ in $K_0(\mathcal{T})$.

Let $\mathcal{T}$ be a category. Denote by $[\mathcal{T}]$ the collection of isomorphism classes of objects in $\mathcal{T}$. Suppose $\mathcal{T}$ is triangulated. Let A and B be subcollections of $[\mathcal{T}]$. Define

$$A * B = \{[Y] \in \mathcal{T} \,|\, \text{there is a distinguished triangle } X \to Y \to Z \rightsquigarrow, \text{ with } [X] \in A \text{ and } [Z] \in B\}.$$

The operation $*$ is associative [BBD, Lemme 1.3.10]. If $\mathcal{A}, \mathcal{B}$ are subcategories of $\mathcal{T}$, define $\mathcal{A} * \mathcal{B}$ to be the full subcategory consisting of objects with isomorphism class in $[\mathcal{A}] * [\mathcal{B}]$. Further, inductively define $\langle \mathcal{A} \rangle_i$, $i \in \mathbb{Z}_{\geq 0}$ by $\langle \mathcal{A} \rangle_0 = 0$ and $\langle \mathcal{A} \rangle_{i+1} = \mathcal{A} * \langle \mathcal{A} \rangle_i$. Since $*$ is associative, it is clear that $\langle \mathcal{A} \rangle_{i+1} = \mathcal{A} * \langle \mathcal{A} \rangle_i = \langle \mathcal{A} \rangle_i * \mathcal{A}$. Set $\langle \mathcal{A} \rangle_\infty = \bigcup_{i>0} \langle \mathcal{A} \rangle_i$.

An object X of a triangulated category is *filtered* by objects $Y_1, \ldots, Y_n$ if there exists a sequence of objects $0 = X_0, X_1, \ldots, X_n = X$ and distinguished triangles $X_{i-1} \to X_i \to Y_i \rightsquigarrow$. The following is immediate from the definitions.

Lemma 1.4.1. *Let $\mathcal{T}$ be a triangulated category and let $\mathcal{A} \subset \mathcal{T}$ be a subcategory. An object $X \in \mathcal{T}$ is filtered by objects in $\mathcal{A}$ if and only if $X \in \langle \mathcal{A} \rangle_\infty$.*

Examples.

(i) *The homotopy category*: Let $\mathcal{A}$ be an additive category. Write $\mathrm{Ho}^*(\mathcal{A})$, $* = \emptyset, -, +, b$, for the homotopy category of complexes, bounded above complexes, bounded below complexes and bounded complexes in $\mathcal{A}$, respectively. Objects of $\mathrm{Ho}(\mathcal{A})$ are complexes in $\mathcal{A}$ and morphisms are chain maps up to homotopy equivalence. The categories $\mathrm{Ho}^*(\mathcal{A})$ are triangulated with the shift functor given by the shift on complexes and with distinguished triangles given by those sequences of morphisms that are isomorphic to standard triangles (1.1).

(ii) *The derived category*: Let $\mathcal{A}$ be an abelian category and let $\mathcal{D}^*(\mathcal{A})$, $* = \emptyset, -, +, b$, denote the derived category, the bounded above derived category, the bounded below derived category and the bounded derived category of $\mathcal{A}$, respectively. The category $\mathcal{D}^*(\mathcal{A})$ is obtained by localizing $\mathrm{Ho}^*(\mathcal{A})$ with respect to the class of morphisms that induce isomorphisms on cohomology (for details on localization we refer the reader to [KaSc, Chapters 7, 10.3, 13]).

Each of the categories $\mathcal{D}^*(\mathcal{A})$, $* = \emptyset, -, +, b$, is triangulated. The shift functor is the shift on complexes and distinguished triangles are those sequences of morphisms that are isomorphic to standard triangles (1.1).

It is useful to note that if $0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$ is an exact sequence in $\mathrm{Comp}(\mathcal{A})$, then there exists a distinguished triangle $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1]$ in $\mathcal{D}(\mathcal{A})$, moreover $Z^\bullet$ is isomorphic to $\mathrm{cone}(f)$ in $\mathcal{D}(\mathcal{A})$ [KaSc, Proposition 13.1.14].

Viewing each object of $\mathcal{A}$ as a complex in degree 0, we regard $\mathcal{A}$ as a subcategory of $\mathcal{D}^*(\mathcal{A})$. This is a full and faithful embedding [KaSc, Proposition 13.1.10(i)]. Furthermore, $\mathrm{Ext}^i(X, Y) = 0$ for all $i < 0$ and each $X, Y \in \mathcal{A}$ [KaSc, Proposition 13.1.10 (ii)]. For interpretations of higher Ext groups see [KaSc, Exercises 13.15, 13.16].

Let $X \in \mathcal{D}(\mathcal{A})$, then it is clear that $X \in \mathcal{A}$ if and only if $H^i(X)$ vanishes for all $i \neq 0$. It follows that $X \in \mathcal{A}$ if and only if X is filtered by objects in $\mathcal{A}$.

1.5. **Abelian categories.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. We write $\mathcal{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B})$ for the category of exact (i.e., preserving short exact sequences) functors from $\mathcal{A}$ to $\mathcal{B}$.

The *Grothendieck group* $K_0(\mathcal{A})$, of an abelian category $\mathcal{A}$, is the free abelian group on symbols $[X]$, $X \in \mathcal{A}$, modulo the relation $[X] = [X_1] + [X_2]$ for each short exact sequence $0 \to X_1 \to X \to X_2 \to 0$.

The map

$$K_0(\mathcal{D}^b(\mathcal{A})) \to K_0(\mathcal{A}),$$
$$[X^\bullet] \mapsto \sum_{i \in \mathbb{Z}} [H^i(X^\bullet)]$$

is a group isomorphism. The inverse is given by the inclusion $K_0(\mathcal{A}) \hookrightarrow K_0(\mathcal{D}^b(\mathcal{A}))$. We identify $K_0(\mathcal{D}^b(\mathcal{A}))$ with $K_0(\mathcal{A})$ via this isomorphism.

Let $\{L_i\}$ be a set of objects in $\mathcal{A}$ such that the classes $[L_i]$ comprise a basis of $K_0(\mathcal{A})$. Then for $M \in \mathcal{A}$, we write $[M : L_i]$ for the coefficient of L_i when $[M]$ is expanded in terms of the basis $\{[L_i]\}$. Note that $[M : L_i] \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{A}$ be an abelian category. A *simple object* is an object $L \in \mathcal{A}$ such that any monomorphism $A \to L$ is either 0 or an isomorphism (this automatically implies that any morphism $L \to A$ is either 0 or a monomorphism). An *object of length one* is synonymous with simple object. For $n \geq 2$, *objects of length* n are inductively defined to be those objects X such that fit into an exact sequence $0 \to X' \to X \to L \to 0$, with X' of length $n-1$ and L simple. Suppose every object in $\mathcal{A}$ has finite length, then the Jordan-Hölder theorem holds in $\mathcal{A}$ (with the usual proof), i.e., for an object $A \in \mathcal{A}$, the length of A is well defined and the simple objects that occur in a 'composition series' of A are unique up to isomorphism and permutation. The category of finite dimensional representations of an algebra is the prototype of such a category.

2. Adjoint functors

2.1. **Adjunctions.** Given categories $\mathcal{A}$ and $\mathcal{B}$, an *adjunction* (F^*, F) is a pair of functors $F : \mathcal{A} \to \mathcal{B}$ and $F^* : \mathcal{B} \to \mathcal{A}$, and two natural transformations $\varepsilon : F^*F \to \mathrm{id}_{\mathcal{A}}$ and $\eta : \mathrm{id}_{\mathcal{B}} \to FF^*$, such that the compositions

$$F \xrightarrow{\eta \mathbb{1}_F} FF^*F \xrightarrow{\mathbb{1}_F \varepsilon} F \qquad \text{and} \qquad F^* \xrightarrow{\mathbb{1}_{F^*}\eta} F^*FF^* \xrightarrow{\varepsilon \mathbb{1}_{F^*}} F^*$$

are equal to the identity on F and F^*, respectively. The morphisms η and ε are the *unit* and *counit* of the adjunction, respectively.

An adjunction gives an isomorphism, functorial in $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$\alpha_{A,B} : \mathrm{Hom}_{\mathcal{A}}(F^*(B), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(B, F(A)),$$
$$f \mapsto \mathbb{1}_F f \circ \eta_B.$$

The inverse is given by $f' \mapsto \varepsilon_A \circ \mathbb{1}_{F^*} f'$.

Conversely, a functorial isomorphism $\alpha_{A,B} : \mathrm{Hom}_{\mathcal{A}}(F^*(B), A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(B, (F(A))$ provides an adjunction (F^*, F). Namely, set

$$\varepsilon_A = \alpha^{-1}_{A,F(A)}(\mathrm{id}_{F(A)}) \qquad \text{and} \qquad \eta_B = \alpha_{F^*(B),B}(\mathrm{id}_{F^*(B)}).$$

If (F^*, F) is an adjunction, then the functor F^* is *left adjoint* to F and the functor F is *right adjoint* to F^*.

The definitions imply:

Lemma 2.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Suppose (F^*, F) is an adjunction between functors $F^* : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$.*

(i) *If $X \in \mathcal{A}$ is such that $F^*(X) \neq 0$, then the unit map $\eta_X : X \to FF^*(X)$ is non-zero.*
(ii) *If $Y \in \mathcal{B}$ is such that $F(Y) \neq 0$, then the counit map $\varepsilon_Y : F^*F(Y) \to Y$ is non-zero.*

2.2. **Transpose maps.** Let (F^*, F) and (G^*, G) be adjunctions between functors $F^*, G^* : \mathcal{A} \to \mathcal{B}$ and $F, G : \mathcal{B} \to \mathcal{A}$. Let η and ε denote the unit and counit of the adjunction (F^*, F), and let η' and ε' denote the unit and counit of the adjunction (G^*, G). Let $\phi \in \mathrm{Hom}(F, G)$. The *transpose* $\phi^* : G^* \to F^*$ is the composition

$$G^* \xrightarrow{\mathbb{1}_{G^*}\eta} G^*FF^* \xrightarrow{\mathbb{1}_{G^*}\phi\mathbb{1}_{F^*}} G^*GF^* \xrightarrow{\varepsilon'\mathbb{1}_{F^*}} F^*. \tag{2.1}$$

Proposition 2.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. Let (F^*, F) and (G^*, G) be adjunctions between functors $F^*, G^* : \mathcal{A} \to \mathcal{B}$ and $F, G : \mathcal{B} \to \mathcal{A}$. Let $\phi : F \to G$ be a natural transformation.*

(i) *Let η, ε denote the unit and counit of (F^*, F) and let η', ε' be the unit and counit of (G^*, G). Then the following diagrams commute:*

$$\begin{tikzcd} F^*F \arrow[r, "\varepsilon"] & \mathrm{id} \\ G^*F \arrow[u, "\phi^*\mathbb{1}_F"] \arrow[r, "\mathbb{1}_{G^*}\phi"'] & G^*G \arrow[u, "\varepsilon'"'] \end{tikzcd} \qquad \begin{tikzcd} FF^* \arrow[r, "\phi\mathbb{1}_{F^*}"] & GF^* \\ \mathrm{id} \arrow[u, "\eta"] \arrow[r, "\eta'"'] & GG^* \arrow[u, "\mathbb{1}_G\phi^*"'] \end{tikzcd}$$

(ii) *If $\psi : F \to G$ is a natural transformation, then $(\phi + \psi)^* = \phi^* + \psi^*$.*
(iii) *Let (H^*, H) be an adjunction between functors $H^* : \mathcal{A} \to \mathcal{B}$ and $H : \mathcal{B} \to \mathcal{A}$. Further, let $\psi : G \to H$ be a natural transformation. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.*

Proof.

(i) For the first diagram we have that

$$\begin{aligned} \varepsilon \circ \phi^*\mathbb{1}_F &= \varepsilon \circ \varepsilon'\mathbb{1}_{F^*}\mathbb{1}_F \circ \mathbb{1}_{G^*}\phi\mathbb{1}_{F^*}\mathbb{1}_F \circ \mathbb{1}^*_G\eta\mathbb{1}_F \\ &= \varepsilon' \circ \mathbb{1}_{G^*}\mathbb{1}_G\varepsilon \circ \mathbb{1}_{G^*}\phi\mathbb{1}_{F^*}\mathbb{1}_F \circ \mathbb{1}^*_G\eta\mathbb{1}_F \\ &= \varepsilon' \circ \mathbb{1}_{G^*}\phi \circ \mathbb{1}_{G^*}\mathbb{1}_F\varepsilon \circ \mathbb{1}_{G^*}\eta\mathbb{1}_F \\ &= \varepsilon' \circ \mathbb{1}_{G^*}\phi. \end{aligned}$$

The first equality is the definition of ϕ^*, the second and third equality are a consequence of ε' and ϕ being morphisms of functors. The last equality follows from the definition of the unit and counit.

The proof for the second diagram is similar.

(ii) Recall that we are assuming that all functors between additive categories are additive, i.e., the induced maps on Hom groups are homomorphisms. The claim is now immediate from the definitions.
(iii) Let η'', ε'' be the unit and counit of (H^*, H). Then

$$\begin{aligned} \phi^* \circ \psi^* &= \varepsilon'\mathbb{1}_{F^*} \circ \mathbb{1}_{G^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{G^*}\eta \circ \varepsilon''\mathbb{1}_{G^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*} \circ \mathbb{1}_{H^*}\eta' \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \mathbb{1}_{G^*}\phi\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*FF^*} \circ \mathbb{1}_{H^*HG^*}\eta \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*} \circ \mathbb{1}_{H^*}\eta' \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \mathbb{1}_{G^*}\phi\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*FF^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*FF^*} \circ \mathbb{1}_{H^*GG^*}\eta \circ \mathbb{1}_{H^*}\eta' \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*HG^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*FF^*} \circ \mathbb{1}_{H^*GG^*}\eta \circ \mathbb{1}_{H^*}\eta' \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*GG^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*GG^*}\eta \circ \mathbb{1}_{H^*}\eta' \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*GG^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta'\mathbb{1}_{FF^*} \circ \mathbb{1}_{H^*}\eta \\ &= \varepsilon'\mathbb{1}_{F^*} \circ \varepsilon''\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*}\eta'\mathbb{1}_{GF^*} \circ \mathbb{1}_{H^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta \\ &= \varepsilon''\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*H}\varepsilon'\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{G^*GF^*} \circ \mathbb{1}_{H^*}\eta'\mathbb{1}_{GF^*} \circ \mathbb{1}_{H^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta \\ &= \varepsilon''\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*G}\varepsilon'\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta'\mathbb{1}_{GF^*} \circ \mathbb{1}_{H^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta \\ &= \varepsilon''\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\psi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\phi\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta \\ &= \varepsilon''\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}(\psi \circ \phi)\mathbb{1}_{F^*} \circ \mathbb{1}_{H^*}\eta \\ &= (\psi \circ \phi)^*. \end{aligned}$$

All of the equalities, except the last three, are due to the fact that all the morphisms involved are natural transformations. The first of the last three equalities follows from the definition of the unit and counit, the remaining two are obvious.

□

Proposition 2.2.2. *Let (F^*, F) be an adjunction between functors $F^* : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$.*

(i) $\mathrm{id}_F^* = \mathrm{id}_{F^*}$*;*
(ii) $0^* = 0$*;*
(iii) *if $e : F \to F$ is idempotent, i.e., $e^2 = e$, then e^* is also idempotent.*

Proof. (i) is immediate from the definition of transpose maps and the defining properties of the unit/counit. (ii) follows from Proposition 2.2.1 (ii). Combining (i),(ii) and Proposition 2.2.1 (ii), (iii), we get that

$$e^* \circ (\mathrm{id}_{F^*} - e^*) = ((\mathrm{id}_F - e) \circ e)^* = 0^* = 0.$$

This shows (iii). □

2.3. **Composing adjoints.** Let (F^*, F) and (G^*, G) be adjunction between functors $G^* : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{A}$, $F^* : \mathcal{B} \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{B}$. Let η and ε be the unit and counit of (F^*, F), and let η' and ε' be the unit and counit of (G^*, G).

Let $\overline{\eta} : \mathrm{id}_{\mathcal{B}} \to GFF^*G^*$ be the composition $\mathrm{id}_{\mathcal{A}} \xrightarrow{\eta'} GG^* \xrightarrow{\mathbb{1}_G \eta \mathbb{1}_{G^*}} GF^*FG^*$, and let $\overline{\varepsilon} : F^*G^*GF \to \mathrm{id}_{\mathcal{A}}$ be the composition $F^*G^*GF \xrightarrow{\mathbb{1}_{F^*}\varepsilon' \mathbb{1}_F} F^*F \xrightarrow{\varepsilon} \mathrm{id}_{\mathcal{B}}$.

Lemma 2.3.1. *The natural transformations $\overline{\eta}$ and $\overline{\varepsilon}$ define an adjunction (F^*G^*, GF).*

Proof. We have

$$\mathbb{1}_{GF}\overline{\varepsilon} \circ \overline{\eta}\mathbb{1}_{GF} = \mathbb{1}_{GF}\varepsilon \circ \mathbb{1}_{GFF^*}\varepsilon'\mathbb{1}_F \circ \mathbb{1}_G\eta\mathbb{1}_{G^*GF} \circ \eta'\mathbb{1}_{GF} = \mathbb{1}_{GF}\varepsilon \circ \mathbb{1}_G\eta\mathbb{1}_F \circ \mathbb{1}_G\varepsilon'\mathbb{1}_F \circ \eta'\mathbb{1}_{GF} = \mathbb{1}_{GF},$$

where the first equality is by definition, the second equality holds due to η and ε' being natural transformations and the last equality follows from the definition of unit/counit.

The proof that $\overline{\varepsilon}\mathbb{1}_{F^*G^*} \circ \mathbb{1}_{F^*G^*}\overline{\eta} = \mathbb{1}_{F^*G^*}$ is similar. □

Lemma 2.3.2.

(i) *The natural transformation η' is the transpose of ε, i.e., $\varepsilon^* = \eta'$.*
(ii) *The natural transformation ε is the transpose of η', i.e., $(\eta')^* = \varepsilon$.*

Proof. By definition,

$$\varepsilon^* = \varepsilon\mathbb{1}_{F^*F} \circ \overline{\eta} = \varepsilon\mathbb{1}_{F^*F} \circ \mathbb{1}_{F^*}\eta\mathbb{1}_F \circ \eta' = \eta'.$$

Similarly,

$$(\eta')^* = \varepsilon \circ \mathbb{1}_{F^*}\varepsilon'\mathbb{1}_F \circ \mathbb{1}_{F^*F}\eta' = \varepsilon.$$

□

2.4. **Right transposes.** Let (F^*, F) and (G^*, G) be adjunctions between functors $F^*, G^* : \mathcal{A} \to \mathcal{B}$ and $F, G : \mathcal{B} \to \mathcal{A}$. Write η and ε for the unit and counit of (F^*, F), and write η' and ε' for the unit and counit of $(G*, G)$. Suppose $\psi : G^* \to F^*$ is a natural transformation. Then the *right transpose* $\psi_* : F \to G$ is the composition

$$F \xrightarrow{\eta'\mathbb{1}_F} GG^*F \xrightarrow{\mathbb{1}_G\psi'\mathbb{1}_F} GF^*F \xrightarrow{\mathbb{1}_G\varepsilon} G.$$

The following allows us to transport all results for transposes to right transposes:

Proposition 2.4.1. *Let (F^*, F) and (G^*, G) be adjunctions between functors $F^*, G^* : \mathcal{A} \to \mathcal{B}$ and $F, G : \mathcal{B} \to \mathcal{A}$. Let $\phi : F \to G$ be a natural transformation. Then $(\phi^*)_* = \phi$. Similarly, if $\psi : G^* \to F^*$ is a natural transformation, then $(\psi_*)^* = \psi$*

Proof. Let η, ε be the unit and counit of (F^*, F) and let η', ε' be the unit and counit of (G^*, G). Then

$$\begin{aligned}
(\phi^*)_* &= \mathbb{1}_G\varepsilon \circ \mathbb{1}_G\varepsilon'\mathbb{1}_{F^*F} \circ \mathbb{1}_{GG^*}\phi\mathbb{1}_{F^*F} \circ \mathbb{1}_{GG^*}\eta\mathbb{1}_F \circ \eta'\mathbb{1}_F \\
&= \mathbb{1}_G\varepsilon \circ \mathbb{1}_G\varepsilon'\mathbb{1}_{F^*F} \circ \mathbb{1}_{GG^*}\phi\mathbb{1}_{F^*F} \circ \eta'\mathbb{1}_{FF^*F} \circ \eta\mathbb{1}_F \\
&= \mathbb{1}_G\varepsilon \circ \mathbb{1}_G\varepsilon'\mathbb{1}_{F^*F} \circ \eta'\mathbb{1}_{GF^*F} \circ \phi\mathbb{1}_{F^*F} \circ \eta\mathbb{1}_F \\
&= \mathbb{1}_G\varepsilon \circ \phi\mathbb{1}_{F^*F} \circ \eta\mathbb{1}_F \\
&= \phi \circ \mathbb{1}_F\varepsilon \circ \eta\mathbb{1}_F \\
&= \phi.
\end{aligned}$$

The first equality is by definition, the second, third and fifth equalities are due to the fact that all morphisms involved are natural transformations. The fourth and last equality follow from the definition of the unit and counit.

The proof that $(\psi_*)^* = \psi$ is similar. ☐

2.5. **Complexes of functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $F_0, F_1, \dots, F_n$ be functors from $\mathcal{A}$ to $\mathcal{B}$. Suppose

$$F = 0 \longrightarrow F_0 \xrightarrow{d'_0} F_1 \longrightarrow \cdots \xrightarrow{d'_{n-1}} F_n \longrightarrow 0$$

is a complex. That is, each $d'_i : F_i \to F_{i+1}$ is a natural transformation and $d'_{i+1} \circ d'_i = 0$. Let $\cdots \longrightarrow X^i \xrightarrow{d''_i} X^{i+1} \longrightarrow \cdots$ be a complex in $\mathcal{A}$. Then we obtain a double complex

$$\begin{tikzcd}
 & {} & {} & \\
\cdots \arrow[r] & F_{i+1}(X^j) \arrow[u] \arrow[r, "F_{i+1}(d''_j)"] & F_{i+1}(X^{j+1}) \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & F_i(X^j) \arrow[u, "d'_i"] \arrow[r, "F_i(d''_j)"'] & F_i(X^{j+1}) \arrow[u, "d'_i"'] \arrow[r] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

This gives a fully faithful embedding of $\mathrm{Comp}^b(\mathcal{H}\mathrm{om}(\mathcal{A}, \mathcal{B}))$ in $\mathcal{H}\mathrm{om}(\mathrm{Comp}(\mathcal{A}), \mathrm{Comp}(\mathrm{Comp}(\mathcal{B})))$. Furthermore, taking the total complex of this double complex allows us to consider F as a functor from $\mathrm{Comp}(\mathcal{A})$ to $\mathrm{Comp}(\mathcal{B})$:

$$\mathrm{Comp}^b(\mathcal{H}\mathrm{om}(\mathcal{A}, \mathcal{B})) \hookrightarrow \mathcal{H}\mathrm{om}(\mathrm{Comp}(\mathcal{A}), \mathrm{Comp}(\mathrm{Comp}(\mathcal{B}))) \xrightarrow{\mathrm{Tot}} \mathcal{H}\mathrm{om}(\mathrm{Comp}(\mathcal{A}), \mathrm{Comp}(\mathcal{B})).$$

Proposition 2.5.1. *Let (F_i^*, F_i), $i = 0, 1, \dots, n$, be adjunctions between functors $F_i^* : \mathcal{A} \to \mathcal{B}$ and $F_i : \mathcal{B} \to \mathcal{A}$. Suppose*

$$\Upsilon = 0 \longrightarrow F_0 \xrightarrow{\phi_0} F_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} F_n \longrightarrow 0,$$

with F_0 in degree 0, is a complex of functors. Set

$$\Upsilon^* = 0 \longrightarrow F_n^* \xrightarrow{\phi_{n-1}^*} F_{n-1}^* \xrightarrow{\phi_{n-2}^*} \cdots \xrightarrow{\phi_0^*} F_0^* \longrightarrow 0,$$

with F_0^* in degree 0. Then Υ^* is left adjoint to Υ.

Proof. The composition $\Upsilon^*\Upsilon$ is given by the double complex

$$\begin{tikzcd}
 & {} & {} & \\
\cdots \arrow[r] & F_i^*F_i \arrow[u] \arrow[r, "\mathbb{1}_{F_i^*}\phi_i"] & F_i^*F_{i+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & F_{i+1}^*F_i \arrow[u, "\phi_i^*\mathbb{1}_{F_i}"] \arrow[r, "\mathbb{1}_{F_{i+1}^*}\phi_i"'] & F_{i+1}^*F_{i+1} \arrow[u, "\phi_i^*\mathbb{1}_{F_{i+1}}"'] \arrow[r] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

with $F_i^*F_j$ in bidegree $(-i, j)$. The degree 0 term of this total complex is $\bigoplus_{i=0}^n F_i^*F_i$. Furthermore, the differential on the degree 0 term, is given by

$$\begin{pmatrix} \phi_{i-1}^*\mathbb{1}_{F_i} \\ (-1)^i\mathbb{1}_{F_i^*}\phi_i \end{pmatrix} : F_i^*F_i \to F_{i-1}^*F_i \oplus F_i^*F_{i+1}.$$

View the identity functor as a complex concentrated in degree 0. Define a map $\mathrm{ev} : \Upsilon^*\Upsilon \to \mathrm{id}$ by

$$\begin{pmatrix} \varepsilon_0 & \varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 & \varepsilon_5 & \cdots \end{pmatrix} : \bigoplus_{i=0}^n F_i^*F_i \to \mathrm{id}.$$

Proposition 2.2.1 (i) implies that this is a chain map.

Similarly $\Upsilon\Upsilon^*$ is given by the double complex

$$\begin{tikzcd}
 & {} & {} & \\
\cdots \arrow[r] & F_iF_i^* \arrow[u] \arrow[r, "\mathbb{1}_{F_i}\phi_{i-1}^*"] & F_iF_{i-1}^* \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & F_{i-1}F_i^* \arrow[u, "\phi_{i-1}\mathbb{1}_{F_{i-1}^*}"] \arrow[r, "\mathbb{1}_{F_{i-1}}\phi_{i-1}^*"'] & F_{i-1}F_{i-1}^* \arrow[u, "\phi_{i-1}\mathbb{1}_{F_{i-1}^*}"'] \arrow[r] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

with $F_iF_j^*$ in bidegree $(i, -i)$. The degree 0 term of this total complex is $\bigoplus_{i=0}^n F_iF_i^*$. Furthermore, the differential on the degree 0 term is given by

$$\begin{pmatrix} \phi_i\mathbb{1}_{F_i^*} \\ (-1)^i\mathbb{1}_{F_i}\phi_{i-1}^* \end{pmatrix} : F_iF_i^* \to F_{i+1}F_i^* \oplus F_iF_{i-1}^*.$$

As before, view the identity functor as a complex concentrated in degree 0 and define a map $\mathrm{coev} : \mathrm{id} \to \Upsilon\Upsilon^*$ by

$$\begin{pmatrix} \eta_0 \\ \eta_1 \\ -\eta_2 \\ -\eta_3 \\ \eta_4 \\ \eta_5 \\ \vdots \end{pmatrix} : \mathrm{id} \mapsto \bigoplus_{i=0}^n F_iF_i^*.$$

Again, it follows from Proposition 2.2.1 (i) that this is a chain map.

As both ev and coev are non-zero only in degree 0, we infer that they give an adjunction (Υ^*, Υ). □

Proposition 2.5.2. *Suppose $F^*, G^* : \mathrm{Comp}(\mathcal{A}) \to \mathrm{Comp}(\mathcal{B})$ and $F, G : \mathrm{Comp}(\mathcal{B}) \to \mathrm{Comp}(\mathcal{A})$ are bounded complexes of functors. Further, suppose that we are given adjunctions (F^*, F) and (G^*, G). Let $\Phi : F \to G$ be a natural transformation. Let $\Upsilon^* = \mathrm{cone}(\Phi^*)[-1]$ and let $\Upsilon = \mathrm{cone}(\Phi)$. Then Υ^* is left adjoint to Υ.*

Proof. Without loss of generality, we may assume that

$$F = 0 \longrightarrow F_0 \xrightarrow{d_0} F_1 \longrightarrow \cdots \xrightarrow{d_{n-1}} F_n \longrightarrow 0 \quad \text{and} \quad G = 0 \longrightarrow G_0 \xrightarrow{d'_0} G_1 \longrightarrow \cdots \xrightarrow{d'_{n-1}} G_n \longrightarrow 0,$$

with F_i, G_i functors from $\mathcal{B}$ to $\mathcal{A}$, and with F_0, G_0 in degree 0. We may further assume that we are given adjunctions (F_i^*, F_i) and (G_i^*, G_i), for each i, such that

$$F^* = 0 \longrightarrow F_n^* \xrightarrow{d^*_{n-1}} F^*_{n-1} \longrightarrow \cdots \xrightarrow{d^*_0} F_0^* \longrightarrow 0 \quad \text{and} \quad G^* = 0 \longrightarrow G_n^* \xrightarrow{d'^*_{n-1}} G^*_{n-1} \longrightarrow \cdots \xrightarrow{d'^*_0} G_0^* \longrightarrow 0,$$

with F_0^*, G_0^* in degree 0, and such that Φ and Φ^* are given by

$$\begin{tikzcd}
\cdots \arrow[r] & G_i \arrow[r, "d'_i"] & G_{i+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & F_i \arrow[r, "d_i"'] \arrow[u, "\phi_i"] & F_{i+1} \arrow[r] \arrow[u, "\phi_{i+1}"'] & \cdots
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\cdots \arrow[r] & F^*_{i+1} \arrow[r, "d^*_i"] & F^*_i \arrow[r] & \cdots \\
\cdots \arrow[r] & G^*_{i+1} \arrow[r, "d'^*_i"'] \arrow[u, "\phi^*_{i+1}"] & G^*_i \arrow[r] \arrow[u, "\phi^*_i"'] & \cdots
\end{tikzcd}$$

respectively. Then we deduce that

$$\Upsilon^i = G_i \oplus F_{i+1} \quad \text{with differential given by} \quad \begin{pmatrix} d'_i & \phi_{i+1} \\ 0 & -d_{i+1} \end{pmatrix} : \Upsilon^i \to \Upsilon^{i+1}.$$

Similarly,

$$(\Upsilon^*)^{-i} = F_i^* \oplus G^*_{i-1} \quad \text{with differential given by} \quad \begin{pmatrix} d^*_{i-1} & \phi^*_{i-1} \\ 0 & -d'^*_{i-2} \end{pmatrix} : (\Upsilon^*)^{-i} \to (\Upsilon^*)^{-(i-1)}.$$

By Proposition 2.5.1, we know that there exists a complex of functors, call it Υ', that is left adjoint to Υ. Moreover, using Proposition 2.2.1 (ii) and (iii) we obtain an explicit description for Υ':

$$(\Upsilon')^{-i} = F^*_{i+1} \oplus G^*_i \quad \text{with differential given by} \quad \begin{pmatrix} -d^*_i & -\phi^*_i \\ 0 & d'^*_{i-1}. \end{pmatrix} : (\Upsilon')^{-i} \to (\Upsilon')^{-(i-1)}.$$

It follows that $\Upsilon^*[-1] = \Upsilon'$. □

Remarks.

(i) Proposition 2.5.2 implies Proposition 2.5.1.

(ii) Let F, F^*, G, G^* and $\phi : F \to G$ be as in the proposition. Consider the induced functors between $\mathrm{Ho}(\mathcal{A})$ and $\mathrm{Ho}(\mathcal{B})$ (or between $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ if $\mathcal{A}$ and $\mathcal{B}$ are abelian). Much of the content of Proposition 2.5.2 is that there are distinguished triangles

$$F \xrightarrow{\phi} G \xrightarrow{\iota} \Upsilon \xrightarrow{\delta} F[1] \quad \text{and} \quad \Upsilon^* \xrightarrow{\iota^*} G^* \xrightarrow{\phi^*} F^* \xrightarrow{\delta^*} \Upsilon^*[1],$$

with Υ^* left adjoint to Υ and where $\iota^*, \delta^*, \phi^*$ signify transpose maps. We note that we haven't yet explained what a 'distinguished triangle of functors' is. For the moment this should be taken to mean that whenever the sequence of natural transformations is evaluated on an object, it yields a distinguished triangle in the corresponding triangulated category. We will clarify this concept in §3.

3. Triangulated functor categories

Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories. By taking total complexes, each object of $\mathrm{Comp}^b(\mathcal{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$ gives an exact functor from $\mathrm{Comp}(\mathcal{A})$ to $\mathrm{Comp}(\mathcal{B})$, see §2.5. This gives a functor

$$\mathrm{Comp}^b(\mathcal{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B})) \to \mathcal{H}om_{\mathfrak{Tr}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B})), \quad * = \emptyset, +, -, b.$$

Write $\mathcal{H}om_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B}))$ for the image of this functor. A moments thought shows that this category is equivalent to $\mathrm{Ho}^b(\mathcal{H}om_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$. In particular, $\mathcal{H}om_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B}))$ is triangulated.

Each $F \in \mathcal{H}om_{\mathfrak{Tr}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B}))$ also induces a functor $\overline{F} : \mathcal{D}^*(\mathcal{A}) \to \mathcal{D}^*(\mathcal{B})$. Thus, we obtain a functor

$$\overline{\cdot} : \mathcal{H}om_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B})) \to \mathcal{H}om_{\mathfrak{Tr}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B})).$$

Let $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ denote the image of this functor. It is clear that $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ is the localization of $\mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$ with respect to the class of all morphisms that become isomorphisms under $\overline{\cdot}$. Our immediate goal is to show that this is a triangulated category. This is essentially a consequence of the fact that distinguished triangles in $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{B}))$ applied to objects of $\mathcal{D}^*(\mathcal{A})$ give distinguished triangles in $\mathcal{D}^*(\mathcal{B})$.

Let $\mathcal{N} \subset \mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$ be the full subcategory consisting of those objects that are sent to the zero functor under $\overline{\cdot}$. Then

Proposition 3.0.3.
(i) $0 \in \mathcal{N}$,
(ii) $F \in \mathcal{N}$ *if and only if* $F[1] \in \mathcal{N}$,
(iii) *if* $F \to G \to H \rightsquigarrow$ *is a distinguished triangle in* $\mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$ *with* $F, H \in \mathcal{N}$, *then* $G \in \mathcal{N}$.

Proof. (i) and (ii) are obvious. For (iii), let $X \in \mathcal{D}^*(\mathcal{A})$. Then $\overline{F}(X) \to \overline{G}(X) \to \overline{H}(X) \rightsquigarrow$ is a distinguished triangle. Thus, $F(X) = H(X) = 0$ implies $G(X) = 0$. □

Define

$$\mathcal{N}\mathcal{Q} = \{\phi : F \to G \,|\, \text{there exists a distinguished triangle } F \xrightarrow{\phi} G \longrightarrow H \rightsquigarrow \text{ with } H \in \mathcal{N}\}.$$

Proposition 3.0.4. *Let* $\mathcal{D}_{\mathcal{N}\mathcal{Q}}$ *be the localization of* $\mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$ *by* $\mathcal{N}\mathcal{Q}$. *Let* $\mathcal{Q} : \mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B})) \to \mathcal{D}_{\mathcal{N}\mathcal{Q}}$ *be the localization functor.*
(i) $\mathcal{D}_{\mathcal{N}\mathcal{Q}}$ *is an additive category endowed with an auto-equivalence (the image of* $[1]$, *still denoted by* $[1]$*).*
(ii) *Define a distinguished triangle in* $\mathcal{D}_{\mathcal{N}\mathcal{Q}}$ *as being isomorphic to the image of a distinguished triangle in* $\mathrm{Ho}^b(\mathcal{A}, \mathcal{B}))$ *under* $\mathcal{Q}$. *Then* $\mathcal{D}_{\mathcal{N}\mathcal{Q}}$ *is a triangulated category and* $\mathcal{Q}$ *is a triangulated functor.*
(iii) *If* $F \in \mathcal{N}$, *then* $\mathcal{Q}(F) = 0$.

Proof. This is a straightforward application of a classical result on localization of triangulated categories, see [KaSc, Theorem 10.2.3]. □

Proposition 3.0.5. *Let* ϕ *be a morphism in* $\mathrm{Ho}^b(\mathcal{H}\mathrm{om}_{\mathfrak{Ab}}(\mathcal{A}, \mathcal{B}))$. *Then* ϕ *is in* $\mathcal{N}\mathcal{Q}$ *if and only if* $\overline{\phi}$ *is an isomorphism in* $\mathcal{H}\mathrm{om}_{\mathfrak{Tr}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$.

Proof. Let $\phi : F \to G$ be in $\mathrm{Ho}^b(\mathcal{H}om(\mathcal{A}, \mathcal{B}))$. Complete ϕ to a distinguished triangle $F \xrightarrow{\phi} G \longrightarrow H \rightsquigarrow$. Then each X in $\mathcal{D}^b(\mathcal{A})$ gives a distinguished triangle

$$\overline{F}(X) \xrightarrow{\overline{\phi}} \overline{G}(X) \longrightarrow \overline{H}(X) \rightsquigarrow .$$

If $\overline{\phi}$ is an isomorphism then $\overline{H}(X) = 0$ and so ϕ belongs to $\mathcal{N}\mathcal{Q}$. Conversely, if ϕ is in $\mathcal{N}\mathcal{Q}$, then the above distinguished triangle reduces to $\overline{F}(X) \xrightarrow{\overline{\phi}} \overline{G}(X) \longrightarrow 0 \rightsquigarrow$. This implies that $\overline{\phi}$ is an isomorphism. □

Corollary 3.0.6. *The category* $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ *is triangulated. Furthermore,*
(i) *The shift functor on* $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ *coincides with composition with* $[1]$;
(ii) *Every distinguished triangle in* $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ *is isomorphic to a standard triangle.*

Proof. Proposition 3.0.5 shows that $\mathcal{D}_{\mathcal{N}\mathcal{Q}}$ is equivalent to $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$. Now Proposition 3.0.4 shows that $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{B}))$ is triangulated and also gives (i) and (ii). □

We have established a triangulated structure on our functor categories. In the situation $\mathcal{A} = \mathcal{B}$, i.e., when dealing with functors from $\mathcal{A}$ to $\mathcal{A}$, there is another important structure on these categories. Namely, composition of functors endows these categories with a monoidal structure. These two structures are compatible in various ways.

Proposition 3.0.7. *Let* $\mathcal{T}$ *be one of* $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{A}))$ *or* $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{A}))$.
(i) *Let* $E \in \mathcal{T}$ *and suppose* $F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{h} F[1]$ *is a distinguished triangle in* $\mathcal{T}$. *Then both*

$$EF \xrightarrow{\mathbb{1}_E f} EG \xrightarrow{\mathbb{1}_E g} EH \xrightarrow{\mathbb{1}_E h} EF[1] \quad \textit{and} \quad FE \longrightarrow GE \longrightarrow HE \longrightarrow FE[1]$$

are distinguished triangles.

(ii) *Let $F \to G \to H \rightsquigarrow$ be a distinguished triangle. If F and G admit left (or right) adjoints, then so does H.*

(iii) *Let $F^*, F, G^*, G, H^*, H \in \mathcal{T}$ and suppose (F^*, F), (G^*, G) and (H^*, H) are adjunctions. Then*

$$F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{h} F[1]$$

is a distinguished triangle if and only if

$$F^*[-1] \xrightarrow{-h^*} H^* \xrightarrow{g^*} G^* \xrightarrow{f^*} F^*$$

is a distinguished triangle.

Proof. (i) is clear, (ii) follows from Proposition 2.5.2. (iii) follows from consideration of standard triangles. □

Remarks.

(i) The results above should generalize easily to the stable category of a Frobenius category. That is, one should be able to put a triangulated structure in much the same way on appropriate categories of functors acting on the stable category. This should encompass 'enhanced triangulated categories', i.e., DG categories etc.

(ii) Perhaps the correct setting for this framework is in the context of the 2-category of Frobenius categories and the 2-category of triangulated categories. Then our results on triangulated functor categories should be a statement involving the existence of a 2-functor between these categories.

(iii) Triangulated functor categories should perhaps be developed under the framework of 'triangulated monoidal categories'.

(iv) If $\mathcal{A}$ is the category of representations of a finite dimensional algebra then one can show that an equivalence in $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{A}))$ lifts to an equivalence in $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathrm{Ho}^*(\mathcal{A}), \mathrm{Ho}^*(\mathcal{A}))$. Can this be generalized to other abelian categories?

(v) In what setting can every equivalence on $\mathcal{D}^*(\mathcal{A})$ be obtained as an equivalence in $\mathcal{H}\mathrm{om}_{\mathrm{alg}}(\mathcal{D}^*(\mathcal{A}), \mathcal{D}^*(\mathcal{A}))$? Some sort of partial answer should be provided by Rickard's Morita theory for triangulated categories. Is the situation analogous to Orlov's result in the geometric context on Fourier-Mukai kernels?

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