THE GROTHENDIECK GROUP OF THE DERIVED CATEGORY

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Let $\mathcal{C}$ be an abelian category, let $\mathcal{D}^b(\mathcal{C})$ be the associated bounded derived category. Denote by $K_0(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$, this is the free abelian group on isomorphism classes $[M]$ of objects in $\mathcal{C}$ modulo the relations $[M] = [N] + [L]$ for every exact sequence $0 \to N \to M \to L \to 0$. Similarly, $K_0(\mathcal{D}^b(\mathcal{C}))$ is defined as the free abelian group on isomorphism classes $[X]$ of complexes in $\mathcal{D}^b(\mathcal{C})$ modulo the relations $[X] = [Y] + [Z]$ for every distinguished triangle $Y \to X \to Z \rightsquigarrow Y[1]$. We are using the convention that for a complex $X = (X^i, d_X^i)$, the complex $X[n]$ is given by $(X[n])^i = X^{n+i}$ and $d^i_{X[n]} = (-1)^n d_X^{i+n}$.

Lemma 0.1. *With notation as above, let X be a complex in $\mathcal{D}^b(\mathcal{C})$ with cohomology concentrated in exactly one degree, then in $K_0(\mathcal{D}^b(\mathcal{C}))$ we have that*

$$[X] + [X[-1]] = 0.$$

Proof. We may assume that X has exactly one non-zero component, say X^i. Let Y be the complex

$$\cdots \to 0 \to 0 \to X^i \xrightarrow{\mathrm{id}} X^i \to 0 \to 0 \to \cdots$$

where the first X^i is in degree i. Then $Y \simeq 0$ in $\mathcal{D}^b(\mathcal{C})$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & X^i & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X^i & \xrightarrow{\mathrm{id}} & X^i & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The mapping cone of the resulting map is isomorphic to $X[-1]$. □

Remark 0.2. The argument in the latter part of the proof of the next proposition shows that the above lemma is in fact true for an abitrary complex X in $\mathcal{D}^b(\mathcal{C})$.

Proposition 0.3. *With notation as before*

$$K_0(\mathcal{C}) \simeq K_0(\mathcal{D}^b(\mathcal{C})).$$

Proof. Let X be a complex in $\mathcal{D}^b(\mathcal{C})$. We define an element in $K_0(\mathcal{C})$ associated to X, the *Euler characteristic* of X as

$$\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(X)].$$

If $X \sim X'$ in $D^b(\mathcal{C})$, then $H^i(X) \sim H^i(X')$, hence $\chi(X) = \chi(X')$. Moreover, if $Y \to X \to Z \rightsquigarrow Y[1]$ is a distinguished triangle in $\mathcal{D}^b(\mathcal{C})$, then in $\mathcal{C}$ we have the long exact sequence of cohomology

$$\cdots \to H^i(Y^i) \to H^i(X^i) \to H^i(Z^i) \to H^{i+1}(Y^{i+1}) \to \cdots$$

which gives that $\chi(X) = \chi(Y) + \chi(Z)$. Thus, we have a well defined group homomorphism

$$\begin{aligned} \alpha : K_0(\mathcal{D}^b(\mathcal{C})) &\to K_0(\mathcal{C}), \\ [X] &\mapsto \chi(X). \end{aligned}$$

Using the canonical embedding, $\iota : \mathcal{C} \to \mathcal{D}^b(\mathcal{C})$ given by viewing an object of $\mathcal{C}$ as a complex with cohomology concentrated in degree 0, we get a group homomorphism

$$\begin{aligned} \beta : K_0(\mathcal{C}) &\to K_0(\mathcal{D}^b(\mathcal{C})), \\ [X] &\mapsto [\iota(X)]. \end{aligned}$$

It is clear that $\alpha \circ \beta = \mathrm{id}_{K_0(\mathcal{C})}$. On the other hand, for any complex $X \in \mathcal{D}^b(X)$, we claim that

$$[X] = \sum_{i \in \mathbb{Z}} (-1)^i [\iota(H^i(X))].$$

This is seen as follows: let X^i be the largest non-zero component of X. Then the following diagram is commutative

$$\begin{array}{ccccccccccc} \longrightarrow & X^{i-3} & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \xrightarrow{\varphi} & \mathrm{im}(\varphi) & \longrightarrow & 0 \\ & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ \longrightarrow & X^{i-3} & \longrightarrow & X^{i-2} & \longrightarrow & X^{i-1} & \xrightarrow{\varphi} & X^i & \longrightarrow & 0 \end{array}$$

The mapping cone of the resulting map is isomorphic to $\iota(H^i(X))[-i]$ in $\mathcal{D}^b(\mathcal{C})$, now using the previous lemma and iterating this construction on the top row of the above diagram we get that $[X] = \sum_{i \in \mathbb{Z}} (-1)^i [\iota(H^i(X))]$. Thus, $\beta \circ \alpha = \mathrm{id}_{K_0(\mathcal{D}^b(\mathcal{C}))}$. □

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