ADJOINT FUNCTORS

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For simplicity, we will assume all categories to be *concrete* categories, i.e., objects have an underlying set structure and morphisms are completely determined by their effect on the underlying sets.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $F^\vee : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is *left adjoint* to $F^\vee$ (or $F^\vee$ is *right adjoint* to F) and that $(F, F^\vee)$ form an adjoint pair if there is a map of sets

$$\alpha : \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, F^\vee(Y))$$

that is functorial in X and Y, $X \in \mathcal{C}$, $Y \in \mathcal{D}$.

Substituting $Y = F(X)$ we obtain a map

$$\eta_X := \alpha(\mathrm{id}_{F(X)}) : X \to F^\vee F(X).$$

Substituting $X = F^\vee(Y)$ we obtain a map

$$\varepsilon_Y := \alpha^{-1}(\mathrm{id}_{F^\vee(Y)}) : FF^\vee(Y) \to Y.$$

The families $\{\eta_X\}$ and $\{\varepsilon_Y\}$ define natural transformations

$$\eta : \mathrm{id}_C \to F^\vee F, \qquad \varepsilon : FF^\vee \to \mathrm{id}_D$$

called the *counit* and *unit*, respectively. Both maps are also called the *adjunction maps*.

Let $f : F(X) \to Y$ be a map in $\mathcal{D}$, then by functoriality the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X, F^\vee F(X)) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (F^\vee f)_*} \\ \mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}}(X, F^\vee(Y)) \end{array}$$

where f_* and $(F^\vee f)_*$ are the induced maps on Hom. In particular

$$\alpha(f) = \alpha(f \circ \mathrm{id}_{F(X)}) = F^\vee(f) \circ \alpha(\mathrm{id}_{F(X)}) = F^\vee(f) \circ \eta_X.$$

Similarly, given a map $g : X \to F^\vee(Y)$ in $\mathcal{D}$, then

$$\alpha^{-1}(g) = \alpha^{-1}(\mathrm{id}_{F^\vee(Y)} \circ g) = \alpha^{-1}(\mathrm{id}_{F^\vee(Y)} \circ F(g) = \varepsilon_Y \circ F(g).$$

It follows that the compositions

$$(0.1) \qquad F(X) \xrightarrow{F(\eta_X)} FF^\vee F(X) \xrightarrow{\varepsilon_{F(X)}} F(X)$$

and

$$F^\vee(Y) \xrightarrow{\eta_{F^\vee(Y)}} F^\vee F F^\vee(Y) \xrightarrow{F^\vee(\varepsilon_Y)} F^\vee(Y) \tag{0.2}$$

are the maps $\mathrm{id}_{F(X)}$ and $\mathrm{id}_{F^\vee(Y)}$ respectively.

The existence of of adjunction maps is equivalent to $(F, F^\vee)$ being an adjoint pair. Namely, let $F : \mathcal{C} \to \mathcal{D}$ and $F^\vee : \mathcal{D} \to \mathcal{C}$ be functors with the additional data of natural transformations $\eta : \mathrm{id}_\mathcal{C} \to F^\vee F$ and $\varepsilon : FF^\vee \to \mathrm{id}_\mathcal{D}$ that satisfy (0.1) and (0.2). Then $(F, F^\vee)$ is an adjoint pair. The isomorphism α is given by $\alpha(f) = F^\vee(f) \circ \eta_X$ and the inverse α^{-1} is given by $\alpha^{-1}(g) = \varepsilon_Y \circ F(g)$.

Let $(F, F^\vee)$ and $(G, G^\vee)$ be adjoint pairs with units η, $\overline{\eta}$ respectively and counits ε, $\overline{\varepsilon}$ respectively. Let $\varphi \in \mathrm{Hom}(F, G)$. Then, we define $\varphi^\vee : G^\vee \to F^\vee$ as the composition

$$\varphi^\vee : \; G^\vee(Y) \xrightarrow{\eta_{G^\vee(Y)}} F^\vee F G^\vee(Y) \xrightarrow{F^\vee(\varphi_{G^\vee(Y)})} F^\vee G G^\vee(Y) \xrightarrow{F^\vee(\overline{\varepsilon}_Y)} F^\vee(Y),$$

$Y \in \mathcal{D}$. This is the unique map making the following diagram commutative, for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$:

$$\begin{array}{ccc} \mathrm{Hom}_\mathcal{D}(F(X), Y) & \xrightarrow{\sim} & \mathrm{Hom}_\mathcal{C}(X, F^\vee(Y)) \\ \varphi^* \uparrow & & \uparrow \varphi^\vee_* \\ \mathrm{Hom}_\mathcal{D}(G(X), Y) & \xrightarrow{\sim} & \mathrm{Hom}_\mathcal{D}(X, G^\vee(Y)) \end{array}$$

Using the construction of $\varphi^\vee$ it follows that if a functor has a left/right adjoint then this adjoint is unique upto isomorphism.

The following gives useful criteria for showing exactness of functors.

Lemma 0.0.1. *Let $\mathcal{C}$ be an abelian category, then a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{C}$, provided that for every $X \in \mathcal{C}$ the following sequence is exact:*

$$\mathrm{Hom}_\mathcal{C}(X, A) \xrightarrow{\alpha_*} \mathrm{Hom}_\mathcal{C}(X, B) \xrightarrow{\beta_*} \mathrm{Hom}_\mathcal{C}(X, C) \ .$$

Proof. Put $X = A$ to get that $\beta \circ \alpha = \beta_* \circ \alpha_*(\mathrm{id}_A) = 0$, so $\mathrm{im}(\alpha) \subseteq \ker(\beta)$. Now put $X = \ker(\beta)$ and let $\iota : \ker(\beta) \to B$ be the inclusion map. Then $\beta_*(\iota) = \beta \circ \iota = 0$, so there exists $\varphi \in \mathrm{Hom}_\mathcal{C}(\ker(\beta), A)$ such that $\alpha \circ \varphi = \alpha^*(\varphi) = \iota$. So $\ker(\beta) \subseteq \mathrm{im}(\alpha)$. □

Proposition 0.0.2. *Let $\mathcal{C}, \mathcal{D}$ be abelian categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor left adjoint to $F^\vee : \mathcal{D} \to \mathcal{C}$. Then F is a right exact functor and $F^\vee$ is a left exact functor.*

Proof. Let $0 \to A \to B \to C$ be exact in $\mathcal{C}$ and let $X \in \mathcal{C}$, then we have the following commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(F(X), A) & \longrightarrow & \mathrm{Hom}(F(X), B) & \longrightarrow & \mathrm{Hom}(F(X), C) \\ & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ 0 \longrightarrow & \mathrm{Hom}(X, F^{\vee}(A)) & \longrightarrow & \mathrm{Hom}(X, F^{\vee}(B)) & \longrightarrow & \mathrm{Hom}(X, F^{\vee}(C)) \end{array}$$

The top row is exact as the Hom functor is left exact, thus the bottom row is also exact. By 0.0.1, $0 \to F^{\vee}(A) \to F^{\vee}(B) \to F^{\vee}(C)$ must be exact. This proves that every right adjoint is left exact. In particular $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ (which is a right adjoint) is left exact, i.e, F is right exact. □

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