# THE BGG CATEGORY $\mathcal{O}$

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0.1. **Reminder on semisimple Lie Algebras.** We will work with an algebraically closed field of characteristic 0 which may as well be assumed to be $\mathbb{C}$.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and an opposite Borel subalgebra $\mathfrak{b}^-$. The intersection $\mathfrak{b} \cap \mathfrak{b}^-$ is a Cartan subalgebra, denoted $\mathfrak{h}$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and $\mathfrak{n}^- = [\mathfrak{b}^-, \mathfrak{b}^-]$, then $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$. That is, it is convenient for us to think of $\mathfrak{h}$ as a quotient of $\mathfrak{b}$, rather than a subalgebr.

The Lie algebra $\mathfrak{g}$ acts on itself by derivations $\mathrm{ad}_x$, where $\mathrm{ad}_x(y) = [x, y]$, for $x, y \in \mathfrak{g}$. The representation given by $x \mapsto \mathrm{ad}_x$ is called the *adjoint representation.* With respect to the adjoint action of $\mathfrak{h}$, we have the so called *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}.$$

The non-zero eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$ are by definition the *roots* of $\mathfrak{g}$, and we will denote the set of roots by $R$. Similarly, the eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{n}$ are by definition the *positive roots* of $\mathfrak{g}$, and we will denote this set by $R^+$. For $\alpha \in R \cup \{0\}$, we will denote by $\mathfrak{g}_\alpha$ the corresponding eigenspace.

**Theorem 0.1.1.** [Dix, 1.10.2]

(i) $\dim(\mathfrak{g}_\alpha) = 1$ *for all* $\alpha \in R$.
(ii) *If* $\alpha, \beta \in R$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.
(iii) *If* $\alpha \in R$, *then* $-\alpha \in R$, *and* $\mathfrak{h}_\alpha = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *is a one dimensional subspace of* $\mathfrak{h}$*; it contains a unique element* $H_\alpha$ *such that* $\alpha(H_\alpha) = 2$.
(iv) *Let* $\alpha \in R$. *If* $E_\alpha \in \mathfrak{g}_\alpha - \{0\}$, *then there exists a unique element* $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ *such that* $[E_\alpha, E_{-\alpha}] = H_\alpha$, *where* $H_\alpha$ *is as above.*
(v) *The elements of* $R$ *generate* $\mathfrak{h}^*$.

A Lie algebra is not an algebra, in the sense that it is not associative. Instead of working with $\mathfrak{g}$, it is frequently convenient to work with the *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$. This is the associative algebra (with 1) generated by $\mathfrak{g}$ and relations $xy - yx = [x, y]$ for all $x, y \in \mathfrak{g}$.

**Theorem 0.1.2** (Poincaré-Birkhoff-Witt, [Dix])**.** *Let* $(x_1, x_2, \ldots x_r)$ *be any ordered basis of* $\mathfrak{g}$. *Then the elements* $x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}$, $m \in \mathbb{Z}_{\geq 0}$, *form a basis of* $U(\mathfrak{g})$.

A basis of $U(\mathfrak{g})$ of the type in the above theorem will be referred to simply as a PBW basis. It follows that

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}),$$

and that $U(\mathfrak{g})$ is a Noetherian integral domain. Pick a PBW basis and for each basis element $x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}$, set $\deg(x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}) = \sum_i m_i$. Set

$$U(\mathfrak{g})_i = \mathbb{C}\text{-span}\{x \mid \deg(x) \leq i\},$$

then the filtration

$$\mathbb{C} = U(\mathfrak{g})_0 \subset U(\mathfrak{g})_1 \subset U(\mathfrak{g})_2 \subset \cdots$$

is independent of the choice of the original basis. This filtration will be referred to as the *PBW filtration.*

$U(\mathfrak{g})$ is a cocommutative Hopf algebra with comultiplication $\Delta$, antipode $S$ and counit $\varepsilon$ given by

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \qquad S(g) = -g, \qquad \varepsilon(g) = 0, \qquad \text{for all } g \in \mathfrak{g}.$$

Via the comultiplication, the adjoint action of $\mathfrak{g}$ on itself induces a unique action on $U(\mathfrak{g})$, i.e.

$$x \cdot y = xy - yx, \qquad \text{for all } x \in \mathfrak{g} \text{ and } y \in U(\mathfrak{g}).$$

(On the right hand side of the above equality we are using the identification of $\mathfrak{g}$ with $U(\mathfrak{g})_1$). We will abuse both language and notation by referring to this action as the adjoint action, and writing $[x, y]$ for $x \in \mathfrak{g}$ acting on $y \in U(\mathfrak{g})$.

The *root lattice* $Q$ is the subgroup of $\mathfrak{h}^*$ given by the set of eigenvalues of $\mathfrak{h}$ acting on $U(\mathfrak{g})$, i.e.

$$Q = \mathbb{Z}\text{-span}\{\alpha \mid \alpha \in R^+\}.$$

We also set

$$Q^+ = \{\sum_i m_i\alpha_i \mid m_i \in \mathbb{Z}_{\geq 0}, \alpha_i \in R^+\},$$

(i.e. the set of eigenvalues of $\mathfrak{h}$ acting on $U(\mathfrak{n})$). For $\lambda, \mu \in \mathfrak{h}^*$ we shall say that $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

By construction, the category of modules over $U(\mathfrak{g})$ (as an associative algebra) is equivalent to the category of $\mathfrak{g}$-modules. This category will be denoted $\mathfrak{g}$-mod.

0.2. **The BGG category $\mathcal{O}$.** The *category* $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-mod, consisting of modules $M$, satisfying:

(i) The action of $\mathfrak{n}$ on $M$ is locally finite, i.e. for every $v \in M$, the subspace $U(\mathfrak{n})\cdot v \subset M$ is finite-dimensional.
(ii) $M$ is finitely generated as a $\mathfrak{g}$-module.
(iii) The action of $\mathfrak{h}$ on $M$ is locally finite and semisimple.

If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact in $\mathfrak{g}$-mod, and any two of the modules $M_i$ satisfy (i) and (ii), then so does the third module. On the other hand, suppose $M_2$ satisfies (iii), since $U(\mathfrak{h})$ is commutative, every finite dimensional $U(\mathfrak{h})$-module is completely reducible, consequently $M_1$ and $M_3$ also satisfy (iii). However, if $M_1$ and $M_3$ satisfy (iii), it is *not* neccesarily true that $M_2$ also satisfies (iii).

Let $\mathcal{O}^\tau$ be the subcategory of $\mathfrak{g}$-mod consisting of objects satisfying the same axioms as $\mathcal{O}$ except with $\mathfrak{n}$ replaced by $\mathfrak{n}^-$. Via the Cartan involution $\tau$, the category $\mathcal{O}^\tau$ is equivalent to $\mathcal{O}$. Occasionally it is also convenient to work in the subcategory $\overline{\mathcal{O}}$ of $\mathfrak{g}$-mod that consists of objects satisfying (ii) and (iii). It is clear that both $\mathcal{O}$ and $\mathcal{O}^\tau$ are subcategories of $\overline{\mathcal{O}}$ and that all categories in question are abelian.

Let $M \in \mathfrak{g}$-mod, and let $\lambda \in \mathfrak{h}^*$. A vector $v \in M$ is called a *weight vector* of weight $\lambda$ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$, i.e. $v$ is a simultaneous eigenvector for all elements of $\mathfrak{h}$. The $\lambda$ *weight space* of $M$ is defined as

$$M_\lambda = \{v \in M \mid hv = \lambda(h)v, \text{ for all } h \in \mathfrak{h}\}.$$

**Proposition 0.2.1.** *If $M \in \mathcal{O}$ then all weight spaces of $M$ are finite dimensional.*

*Proof.* As $\mathfrak{h}$ acts semisimply on $M$ and the latter is finitely generated, we may assume that $M$ is generated by a finite set of weight vectors. By the PBW theorem we have that $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$. Applying $U(\mathfrak{n})$ to a weight vector of say, weight $\lambda$, we get a finite dimensional vector space $V$ spanned by weight vectors having weights of the form ($\lambda$ + sum of positive roots). The vector space $V$ is stable under $\mathfrak{h}$, while the action of $U(\mathfrak{n}^-)$ on $V$ produces only weights lower than these. Furthermore, only a finite number of elements (standard basis monomials $y_1^{i_1} \cdots y_m^{i_m}$) in $U(\mathfrak{n}^-)$ can yield the same weight when applied to a weight vector in $V$. □

Let $\lambda \in \mathfrak{h}^*$. The *Verma module* $M(\lambda) \in \mathfrak{g}$-mod is defined by the following universal property. For any object $M \in \mathfrak{g}$-mod,

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M) = \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}_\lambda, M),$$

where $\mathbb{C}_\lambda$ is the 1-dimensional $\mathfrak{b}$-module, on which $\mathfrak{b}$ acts through the character

$$\mathfrak{b} \longrightarrow \mathfrak{b}/\mathfrak{n} \xrightarrow{\lambda} \mathbb{C}.$$

By construction, $M(\lambda) \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. From the PBW theorem it is clear that $1 \otimes 1$ freely generates $M(\lambda)$ over $\mathfrak{n}^-$; i.e. the action of $\mathfrak{n}^-$ on $1 \otimes 1$ defines an isomorphism (of $\mathfrak{n}$-modules) $U(\mathfrak{n}^-) \simeq M(\lambda)$. We will write $v_\lambda^+$ for the image of $1 \otimes 1$ in $M(\lambda)$.

**Lemma 0.2.2.** *Verma modules belong to the category $\mathcal{O}$.*

*Proof.* We only need to check that $\mathfrak{n}$ acts locally finitely on $M(\lambda)$. Let $x \in \mathfrak{g}_\alpha$, then $x \cdot M(\lambda)_\mu \subseteq M(\lambda)_{\mu+\alpha}$. Thus, if we let $U(\mathfrak{g})_i$ be the $i$-th term of the PBW filtration on $U(\mathfrak{g})$. It suffices to check that the finite dimensional subspace $U(\mathfrak{g})_i \cdot v_\lambda^+ \subset M(\lambda)$ is $\mathfrak{n}$-stable. For $u \in U(\mathfrak{g})_i$ and $x \in \mathfrak{g}$ we have:

$$x \cdot (u \cdot v_\lambda^+) = u \cdot (x \cdot v_\lambda^+) + [x, u] \cdot v_\lambda^+,$$

where the first term is 0 if $x \in \mathfrak{n}$. Hence, our assertion follows from the fact that $[\mathfrak{g}, U(\mathfrak{g})_i] \subseteq U(\mathfrak{g})_i$. □

Let $V$ be a $\mathfrak{g}$-module. A nonzero vector $v_\lambda^+$ in $V$ is called a *highest weight vector* of weight $\lambda \in \mathfrak{h}^*$ if $h \cdot v_\lambda^+ = \lambda(h)v^+$ for $h \in \mathfrak{h}$ and $\mathfrak{n} \cdot v_\lambda^+ = 0$. Furthermore we say that $V$ is a *highest weight module* if $V = U(\mathfrak{g}) \cdot v_\lambda^+$. By definition, Verma modules are highest weight modules. It is a formal consequence of the definitions that

**Proposition 0.2.3.** *If $V(\lambda)$ is a highest weight module of weight $\lambda$ then $V(\lambda)$ is a quotient of $M(\lambda)$.*

**Corollary 0.2.4.** *Highest weight modules are in category $\mathcal{O}$.*

**Lemma 0.2.5.** *A Verma module $M(\lambda)$ with highest weight vector $v_\lambda^+$ contains a unique maximal submodule and thus admits a unique simple quotient. Furthermore, $M(\lambda)$ is indecomposable.*

*Proof.* Let $S$ be the sum of all proper submodules of $M(\lambda)$. As no proper submodule of $M(\lambda)$ contains the $M(\lambda)_\lambda$ weight space, it is straightforward to check that their sum $S$ does not contain this weight space, i.e.$v_\lambda^+ \notin S$. Thus, $S \neq M(\lambda)$ and $S$ is the required unique maximal submodule of $M(\lambda)$. Furthermore, $M(\lambda)$ cannot be the direct sum of two proper submodules, since each of these is contained in $S$. □

**Proposition 0.2.6.** *Suppose $M$ is a non-zero module in $\mathcal{O}$. Then $M$ has a finite filtration*

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

*such that $M_{i+1}/M_i$ is a highest weight module.*

*Proof.* Observe that $V = U(\mathfrak{n})M$ is finite dimensional. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$ then $M$ itself is a highest weight module. So assume the statement is true for $\dim(V) < n$. Choose $v \in V$ such that the weight of $v$ is maximal amongst all weights in $V$. Let $M_1 = U(\mathfrak{g})v$, then $\overline{M} = M/M_1$ is in $\mathcal{O}$. Furthermore, $\dim(\overline{V}) < \dim(V)$, so we may apply the inductive hypothesis to $\overline{M}$ to obtain the desired filtration. □

**Corollary 0.2.7.** *Every simple module in $\mathcal{O}$ is isomorphic to a module $L(\lambda)$ with $\lambda \in \mathfrak{h}^*$ and is therefore determined uniquely up to isomorphism by its highest weight.*

## References


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