

Categories and Functors

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1 Categories

A *category* \mathcal{C} consists of a collection of *objects* $\text{Ob}(\mathcal{C})$; and for any two objects $A, B \in \text{Ob}(\mathcal{C})$ a set $\text{Hom}(A, B)$ called the set of *morphisms* of A into B ; and for any three objects $A, B, C \in \text{Ob}(\mathcal{C})$ a law of composition (i.e. a map)

$$\text{Hom}(B, C) \times \text{Hom}(A, B) \longrightarrow \text{Hom}(A, C)$$

satisfying the following axioms

1. Two sets $\text{Hom}(A, B)$ and $\text{Hom}(A', B')$ are disjoint unless $A = A'$ and $B = B'$ in which case they are equal.
2. For each object A of \mathcal{C} there is a morphism $\text{id}_A \in \text{Hom}(A, A)$ which acts as left and right identity for the elements of $\text{Hom}(B, A)$ and $\text{Hom}(A, B)$ respectively, for all objects B in \mathcal{C} .
3. The law of composition is associative (when defined), i.e. given $f \in \text{Hom}(A, B), g \in \text{Hom}(B, C), h \in \text{Hom}(C, D)$ then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

for all objects A, B, C, D of \mathcal{C} .

A morphism $f : A \longrightarrow B$ is called an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$. If $A = B$ we also say that the isomorphism is an *automorphism*. A morphism of an object A into itself is called an *endomorphism*. For an object A in \mathcal{C} we denote by $\text{Aut}(A)$ the set of automorphisms of A , and by $\text{End}(A)$ the set of endomorphisms of A .

Let G be a group and let \mathcal{C} be a category and $A \in \text{Ob}(\mathcal{C})$. By an *operation* of G on A we shall mean a homomorphism

$$\varphi : G \longrightarrow \text{Aut}(A).$$

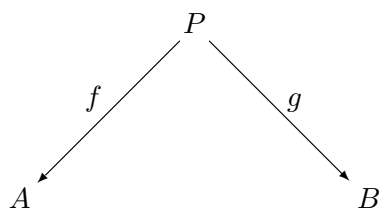
An operation of a group G on an object A is also called a *representation* of G on A .

2 Universal Objects

Let \mathcal{C} be a category. An object U of \mathcal{C} is called *universally attracting* if there exists a unique morphism of each object of \mathcal{C} into U , and is called *universally repelling* if for every object of \mathcal{C} there exists a unique morphism of U into that object. Since a universal object admits the identity morphism into itself, it is clear that if U and U' are two universal objects in \mathcal{C} , then there exists a unique isomorphism between them.

3 Products and Coproducts

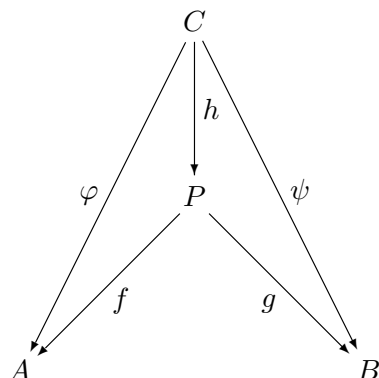
Let \mathcal{C} be a category and let A, B be objects of \mathcal{C} . By a *product* of A, B in \mathcal{C} one means a triple (P, f, g) consisting of an object P in \mathcal{C} and two morphisms f, g



satisfying the following condition. Given two morphisms

$$\varphi : C \longrightarrow A \quad \text{and} \quad \psi : C \longrightarrow B$$

in \mathcal{C} , there exists a unique morphism $h : C \longrightarrow P$ which makes the following diagram commutative



More generally, given a family of objects $\{A_i\}_{i \in I}$ in \mathcal{C} , a product for this family consists of $\{P, \{f_i\}_{i \in I}\}$ where P is an object in \mathcal{C} and $\{f_i\}_{i \in I}$ is a family of morphisms

$$f_i : P \longrightarrow A_i,$$

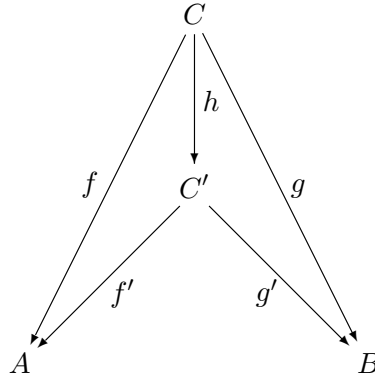
satisfying the following condition; given a family of morphisms

$$g_i : C \longrightarrow A_i,$$

there exists a unique morphism $h : C \longrightarrow P$ such that $f_i \circ h = g_i$ for all i .

Let A, B be objects of a category \mathcal{C} . We note that the product of A, B is universal in the category whose objects consist of pairs of morphisms $f : C \longrightarrow A$ and $g : C \longrightarrow B$ in \mathcal{C} , and whose morphisms are described as follows. Let $f' : C' \longrightarrow A$ and $g' : C' \longrightarrow B$ be another pair,

then a morphism from the first pair to the second is a morphism $h : C \longrightarrow C'$ in \mathcal{C} , making the following diagram commutative



The situation is similar for the product of a family $\{A_i\}_{i \in I}$.

Example. Let \mathcal{C} be the category of sets and let $\{A_i\}_{i \in I}$ be a family of sets. Let $A = \prod_{i \in I} A_i$ be their Cartesian product and let $p_i : A \longrightarrow A_i$ be the projection on the i th factor. Then $(A, \{P_i\}_{i \in I})$ satisfy the requirements of a product in the category of sets.

Example. Let $\{G_i\}_{i \in I}$ be a family of groups, and let $G = \prod G_i$ be their direct product. Let $p_i : G \longrightarrow G_i$ be the projection homomorphism. Then these constitute a product of the family in the category of groups.

Let $\{A_i\}_{i \in I}$ be a family of objects in a category \mathcal{C} . By their *coproduct* one means a pair $\{S, \{f_i\}_{i \in I}\}$ consisting of an object S and a family of morphisms

$$f_i : A_i \longrightarrow S,$$

satisfying the following property. Given a family of morphisms $\{g_i : A_i \longrightarrow C\}$ there exists a unique morphism $h : S \longrightarrow C$ such that $h \circ f_i = g_i$ for all i . The coproduct of a family $\{A_i\}$ will also be denoted by $\coprod A_i$ and similarly the coproduct of two objects A, B will also be denoted by $A \coprod B$. The coproduct of A and B is universal in the category of families maps from A and B into a single object.

Example. Let \mathcal{S} be the category of sets. Then coproducts exist in this category. Let S and S' be sets. Let T be a set having the same cardinality as S' and disjoint from S . Let $f_1 : S \longrightarrow S$ be the identity, and $f_2 : S' \longrightarrow T$ be a bijection. Let U be the union of S and T . Then (U, f_1, f_2) is a coproduct for S and S' .

Example. Let $c\mathcal{S}_0$ be the category of pointed sets. Its objects consist of pairs (S, x) where S is a set and x an element of S . A morphism of (S, x) into (S', x') in this category is a map $g : S \longrightarrow S'$ such that $g(x) = x'$. The coproduct in this category can be constructed as follows. Let T be a set with the same cardinality as S' and such that $T \cap S = \{x\}$. Let $U = S \cup T$, and let

$$f_1 : (S, x) \longrightarrow (U, x)$$

be the map which induces the identity on S . Let

$$f_2 : (S', x') \longrightarrow (U, x)$$

be the map sending x' to x and inducing a bijection of $S' - \{x'\}$ on $T - \{x\}$. Then the triple (U, f_1, f_2) is a coproduct for (S, x) and (S', x') in \mathcal{S}_0 .

4 Fiber products, coproducts, pull-backs and push-outs

Let \mathcal{C} be a category. Let Z be an object of \mathcal{C} . Then we have a new category of objects over Z , denoted by \mathcal{C}_Z . The objects of \mathcal{C}_Z are morphisms:

$$f : X \longrightarrow Z \quad \text{in } \mathcal{C}.$$

A morphism from f to $g : Y \longrightarrow Z$ in \mathcal{C}_Z is merely a morphism $h : X \longrightarrow Y$ in \mathcal{C} which makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & & Z \end{array}$$

A product in \mathcal{C}_Z is called the *fiber product* of f and g in \mathcal{C} and is denoted by $f \times_Z g$, together with its natural morphisms of X, Y over Z , which are sometimes not denoted by anything, but which we denote by p_1 and p_2 .

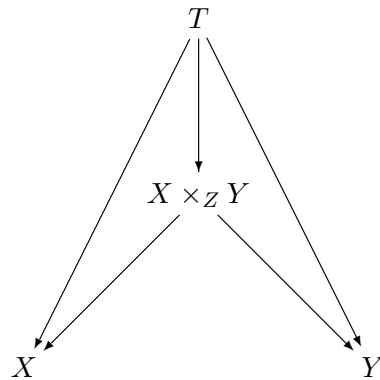
$$\begin{array}{ccc} & X \times_Z Y & \\ p_1 \swarrow & & \searrow p_2 \\ X & & Y \\ & \searrow f & \swarrow g \\ & & Z \end{array}$$

Example. Fiber products exist in the category of abelian groups. The fibered product of two homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ is the subgroup of $X \times Y$ consisting of all pairs (x, y) such that $f(x) = g(y)$.

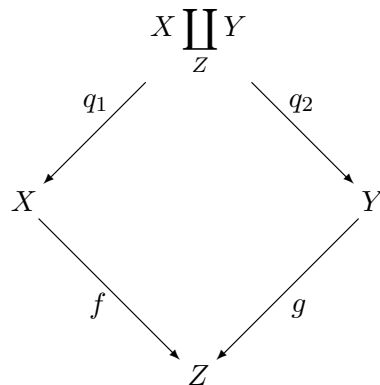
In the fiber product diagram, one calls p_1 , the *pull-back* of g by f , and p_2 the pull-back of f by g . The fiber product satisfies the following universal property: Given any object T in \mathcal{C} and morphisms making the following diagram commutative

$$\begin{array}{ccc} & T & \\ & \swarrow & \searrow \\ X & & Y \\ & \searrow f & \swarrow g \\ & & Z \end{array}$$

there exists a unique morphism $T \longrightarrow X \times_Z Y$ making the following diagram commutative



Dually we have the notion of *coproduct* in the category of morphisms $f : Z \longrightarrow X$ with a fixed object Z as the object of departure of the morphisms. This category is denoted by \mathcal{C}^Z . We reverse the arrows in the preceding discussion. Given two objects f and $g : Z \longrightarrow Y$ in \mathcal{C}^Z we have their coproduct $X \coprod_Z Y$ with morphisms q_1 and q_2 making the following diagram commutative



satisfying the dual universal property of the fiber product. We call it the *fibred coproduct*. We call q_1 the *push-out* of f by f , and q_2 the push-out of f by g .

Example. Fibred coproducts exist in the category of abelian groups. The coproduct of two homomorphisms $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ is the factor group $X \oplus Y/W$ where W is the subgroup of $X \oplus Y$ consisting of all elements $(f(x), -g(z))$ with $z \in Z$.

5 Functors

Let \mathcal{C}, \mathcal{D} be categories. A *covariant functor* F of \mathcal{C} into \mathcal{D} is a rule which to each object X in \mathcal{C} associates an object $F(X)$ in \mathcal{D} , and to each morphism $f : X \longrightarrow Y$ associates a morphism $F(f) : F(X) \longrightarrow F(Y)$ such that:

1. For all X in \mathcal{C} we have $F(\text{id}_X) = \text{id}_{F(X)}$.
2. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are two morphisms of \mathcal{C} then

$$F(g \circ f) = F(g) \circ F(f).$$

We also have the notion of a *contravariant functor* that has essentially the same definition but ‘reverses all arrows’, i.e. to each morphism $f : X \rightarrow Y$ the contravariant functor associates a morphism

$$F(f) : F(B) \rightarrow F(A)$$

going in the opposite direction, such that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms in \mathcal{C} , then

$$F(g \circ f) = F(f) \circ F(g).$$

Example. To each group G associate its set (stripped of the group structure) to obtain a covariant functor from the category of groups into the category of sets, provided we associate with each group homomorphism itself, viewed only as a set theoretic map. Such a functor is called a *stripping* functor or *forgetful* functor.

Example. Consider the category of abelian groups. Fix an abelian group G . The association $X \mapsto \text{Hom}(X, G)$ is a contravariant functor from this category to itself. The association $X \mapsto \text{Hom}(G, X)$ is a covariant functor from the category to itself.

Example. Let \mathcal{C} be a category and G a fixed object in \mathcal{C} . Let $M_G(X) = \text{Hom}(G, X)$ for any object X of \mathcal{C} . If $\varphi : X \rightarrow X'$ is a morphism, let

$$\begin{array}{ccc} M_G(\varphi) : \text{Hom}(G, X) & \longrightarrow & \text{Hom}(G, X'), \\ & g & \longmapsto \varphi \circ g \end{array}$$

for any $g \in \text{Hom}(G, X)$. So M_G is a covariant functor from \mathcal{C} to the category of sets.

Similarly, for each object Y of \mathcal{C} we have a contravariant functor M^G from \mathcal{C} to the category of sets given by $M^G(Y) = \text{Hom}(Y, G)$. M_G and M^G are called *representation functors*.

Let \mathcal{C}, \mathcal{D} be two categories. The functors of \mathcal{C} into \mathcal{D} (say covariant, and in one variable) may be viewed as objects of a category, whose morphisms (called natural transformations) are given as follows. Let F, G be two such functors. A *natural transformation* $H : F \rightarrow G$ is a rule which to each object X of \mathcal{C} associates a morphism

$$H_X : F(X) \rightarrow G(X)$$

such that for any morphism $f : X \rightarrow Y$ the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{H_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{H_Y} & G(Y) \end{array}$$

We therefore have the notion of isomorphisms of functors. A functor is *representable* if it is isomorphic to a representation functor.

6 Adjoint functors

If $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ are functors, then we say that F is a *left adjoint* for G (equivalently G is a *right adjoint* for F) if there is a natural isomorphism $\alpha : \text{Hom}_{\mathcal{C}}(-, G(-)) \cong$

$\text{Hom}_{\mathcal{D}}(F(-), -)$. That is, for every pair of objects X of \mathcal{C} and Y of \mathcal{D} there is a bijection $\alpha_{X,Y} : \text{Hom}_{\mathcal{C}}(X, G(Y)) \cong \text{Hom}_{\mathcal{D}}(F(X), Y)$ such that for every morphism of objects $\varphi : X \rightarrow X'$ in \mathcal{C} and $\psi : Y \rightarrow Y'$ in \mathcal{D} the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, G(Y)) & \xrightleftharpoons{\alpha_{X,Y}} & \text{Hom}_{\mathcal{D}}(F(X), Y) \\ \downarrow (\varphi, G(\psi)) & & \downarrow (F(\varphi), \psi) \\ \text{Hom}_{\mathcal{C}}(X', G(Y)) & \xrightleftharpoons{\alpha_{X',Y'}} & \text{Hom}_{\mathcal{D}}(F(X'), Y') \end{array}$$

Suppose F and F' are left adjoints to $G : \mathcal{D} \rightarrow \mathcal{C}$. For X in \mathcal{C} let $\varphi : F(X) \rightarrow F'(X)$ be the image of $\text{id}_{F'(X)}$ under the adjointness isomorphisms

$$\text{Hom}(F(X), F'(X)) \cong \text{Hom}(X, GF'(X)) \cong \text{Hom}(F'(X), F'(X)).$$

Define $\psi : F'(X) \rightarrow F(X)$ similarly. It can be verified that φ and ψ are natural isomorphisms. Thus, any two left adjoints of G are naturally isomorphic, the same reasoning gives us that any two right adjoints of G are also naturally isomorphic.

Example. Let A be a commutative ring and let be any A -module. The functor $N \mapsto M \otimes_A N$ from $A\text{-mod}$ to itself is the left adjoint of the functor $N \mapsto \text{Hom}_A(M, N)$.

Example. Let U be the functor that takes a Lie algebra to its universal enveloping algebra and let L be the functor that takes an associative algebra to its Lie algebra. Then U is the left adjoint to L .

If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a left adjoint to $G : \mathcal{D} \rightarrow \mathcal{C}$, then for each object Y of \mathcal{D} we have that

$$\text{Hom}_{\mathcal{C}}(F(G(Y)), Y) \cong \text{Hom}_{\mathcal{D}}(G(Y), G(Y)).$$

Let $\varepsilon_Y : F(G(Y)) \rightarrow Y$ be the image of the identity morphism $G(Y) \rightarrow G(Y)$, then ε_B is called the *counit*, and similarly for each object X of \mathcal{C} we get a morphism $\eta_X : X \rightarrow G(F(X))$ called the *unit* of the adjoint pair. The adjointness isomorphism α may be recovered from ε and η . Given a map $\varphi : F(X) \rightarrow Y$ $G(\varphi) \circ \eta_X$ gives us the corresponding map $X \rightarrow G(Y)$. Similarly, given a map $\psi : X \rightarrow G(Y)$ the corresponding map $F(X) \rightarrow Y$ is given by $\varepsilon \circ F(\psi)$.

7 Direct and Inverse Limits

Let I be a set of indices with a partial ordering. We say that I is *directed* if given $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let $I = \{i\}$ be a directed system of indices. Let \mathcal{C} be a category and $\{X_i\}$ a family of objects in \mathcal{C} . For each pair i, j such that $i \leq j$ assume given a morphism

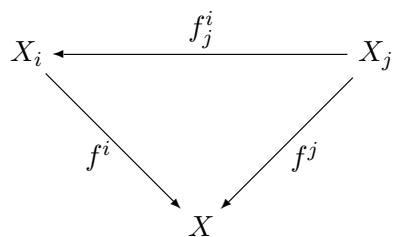
$$f_j^i : X_i \rightarrow X_j$$

such that whenever $i \leq j \leq k$

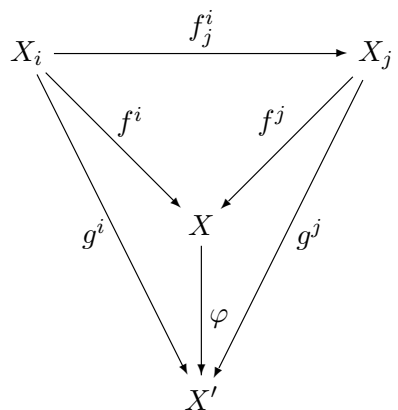
$$f_k^j \circ f_j^i = f_k^i$$

and $f_i^i = \text{id}$. Such a family is called a *directed family of morphisms*. A direct limit for the family f_j^i is a universal object in the following category. The objects consist of pairs $(X, (f^i))$ where

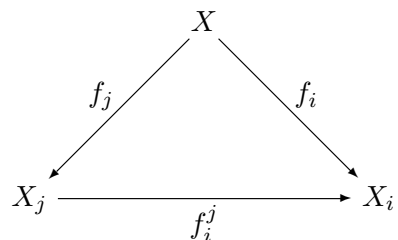
$X \in \text{Ob}(\mathcal{C})$ and (f^i) is a family of morphisms $f^i : X_i \longrightarrow X$, such that for all $i \leq j$ the following diagram commutes



Thus if $(X, (f^i))$ is the direct limit and if $(X', (g^i))$ is any object in the above category, then there exists a unique morphism $\varphi : X \longrightarrow X'$ making the following diagram commute



For simplicity we write $A = \varinjlim A_i$. Reversing the arrows we define inverse limits. Given a directed set I and a family of objects X_i . If $j \geq i$ we are now given a morphism $f_j^i : X_j \longrightarrow X_i$ satisfying $f_k^i \circ f_k^j = f_k^j$ and $f_i^i = \text{id}$, if $j \geq i \geq k$. We now define a category of objects $(X, (f_i))$ with $f_i : X \longrightarrow X_i$ such that the following diagram commutes



A universal object in this category is called an *inverse limit* and we say

$$X = \varprojlim X_i.$$

References

[La] Serge Lang, *Algebra, Revised Third Edition*, Graduate Texts in Mathematics, vol. **211**, Springer-Verlag, Cambridge, 2003. MR1976581.