THE DEGENERATE AFFINE BRAID GROUP AND SCHUR-WEYL TYPE DUALITY

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1. The classical Lie algebras

Let V be an n-dimensional vector space. Let $\mathfrak{gl}_n = \mathfrak{gl}(V)$ denote the Lie algebra of $n \times n$ matrices acting on V. Let $\mathfrak{t}$ denote the Lie subalgebra of $\mathfrak{gl}_n$ consisting of diagonal matrices and let h_i be the matrix with 1 in the i^{th} diagonal entry and 0 elsewhere. Define elements ε_i, in the dual space $\mathfrak{t}^*$, by $\langle \varepsilon_i, h_j \rangle = \delta_{ij}$. The *trace form* $(\cdot|\cdot)$ on $\mathfrak{gl}_n$ is given by $(x|y) = \operatorname{tr}(xy)$, where tr is the ordinary matrix trace. This form is symmetric, ad-invariant and non-degenerate.

Let $\mathfrak{g}$ be one of the classical Lie algebras: $\mathfrak{gl}_n$, $\mathfrak{so}_{2n+1} \subset \mathfrak{gl}_{2n+1}$, $\mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$ or $\mathfrak{so}_{2n} \subset \mathfrak{gl}_{2n}$. For each of these Lie algebras $\mathfrak{g}$, let $\mathfrak{h} \subset \mathfrak{g}$ denote the Lie subalgebra consisting of matrices with non-zero entries only on the diagonal and let $\mathfrak{n} \subset \mathfrak{g}$ denote the Lie subalgebra consisting of matrices with non-zero entries only above the diagonal. Let $\mathfrak{h}^*$ denote the dual space to $\mathfrak{h}$. The set of roots $R \subset \mathfrak{h}^*$ is the set of eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{g}$. The positive roots R^+ are the eigenvalues of $\mathfrak{h}$ acting on $\mathfrak{n}$. We have

$$R^+ = \begin{cases} \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}, & \text{for } \mathfrak{gl}_n, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{so}_{2n+1}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{2\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{sp}_{2n}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}, & \text{for } \mathfrak{so}_{2n}, \end{cases}$$

The trace form descends to an ad-invariant, non-degenerate form on $\mathfrak{g}$. Furthermore, its restriction to $\mathfrak{h}$ is non-degenerate. This gives an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$, $h \mapsto (\cdot, h)$. Via this isomorphism we obtain a non-degenerate form, also denoted $(\cdot|\cdot)$, on $\mathfrak{h}^*$. The form on $\mathfrak{h}^*$ given by $(\varepsilon_i|\varepsilon_j) = \delta_{ij}$.

For each $\alpha \in R$, the coroot $\alpha^\vee \in \mathfrak{h}$ is defined by $\langle \cdot, \alpha^\vee \rangle = \frac{2(\cdot|\alpha)}{(\alpha|\alpha)}$, where $\langle \cdot, \cdot \rangle$ is the evaluation pairing. For each $\alpha \in R$, let $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ be the reflection $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. The Weyl group W_0 is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the reflections s_α, $\alpha \in R$. It is clear that $(\cdot|\cdot)$ is W_0 invariant.

Define $\rho \in \mathfrak{h}^*$ by

$$2\rho = \sum_{i=1}^{n} (y - 2i + 1)\varepsilon_i, \qquad \text{where} \quad y = \begin{cases} 2n-1, & \text{for } \mathfrak{gl}_n, \\ 2n, & \text{for } \mathfrak{so}_{2n+1}, \\ 2n+1, & \text{for } \mathfrak{sp}_{2n}, \\ 2n-1, & \text{for } \mathfrak{so}_{2n}. \end{cases} \tag{1.1}$$

The *dot-action* of W_0 on $\mathfrak{h}^*$ is given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, $w \in W_0$.

The dominant integral weights P^+ are

$$\begin{array}{lll} \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, & \text{for } \mathfrak{gl}_n, \\ & \lambda_1, \ldots, \lambda_r \in \mathbb{Z}, & \\ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, & \\ & \lambda_1, \ldots, \lambda_n \in \mathbb{Z}, \text{ or} & \text{for } \mathfrak{so}_{2n+1}, \\ & \lambda_1, \ldots, \lambda_n \in \frac{1}{2} + \mathbb{Z}, & \\ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, & \text{for } \mathfrak{sp}_{2n}, \\ & \lambda_1, \ldots, \lambda_n \in \mathbb{Z}, & \\ \lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, & \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \geq 0, & \\ & \lambda_1, \ldots, \lambda_n \in \mathbb{Z}, \text{ or} & \text{for } \mathfrak{so}_{2n}, \\ & \lambda_1, \ldots, \lambda_n \in \frac{1}{2} + \mathbb{Z}, & \end{array}$$

where $|\lambda| = \lambda_1 + \cdots + \lambda_n$.

2. Degenerate structures

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\mathfrak{Z}$ denote the center of $U(\mathfrak{g})$. Let $\{b_i\}_i$, $\{b^i\}_i$ be dual bases in $\mathfrak{g}$ with respect to the trace form. The *Casimir* is

$$\kappa = \sum_i b_i b^i.$$

The Casimir is an element in $\mathfrak{Z}$. For a $\mathfrak{g}$-module M, we will write κ_M for the action of κ on M. Note that if M is a highest weight $\mathfrak{g}$-module of weight λ, then κ_M is multiplication by $(\lambda|\lambda + 2\rho)$.

Let Δ denote the coproduct on $U(\mathfrak{g})$. Then

$$\Delta(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa + 2\sum_i b_i \otimes b^i.$$

If $L(\mu)$ and $L(\nu)$ are finite dimensional, highest weight, simple modules of weight μ and ν respectively, then it follows that $2\sum_i b_i \otimes b^i$ acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ as multiplication by

$$(\lambda|\lambda + 2\rho) - (\mu|\mu + 2\rho) - (\nu|\nu + 2\rho).$$

2.1. **The degenerate affine braid group.** The *degenerate affine braid group* $\partial\mathrm{Br}_k$ is the algebra over $\mathfrak{Z}$ presented by generators $t_1, \ldots t_{k-1}$, $e_1, \ldots, e_{k-1}$ and $x_1, \ldots, x_k$, subject to the relations

$$t_i^2 = 1, \tag{2.1}$$
$$t_i t_j = t_j t_i, \qquad \text{if } |i-j| > 1, \tag{2.2}$$
$$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \tag{2.3}$$
$$t_i e_j = e_j t_i, \qquad \text{if } |i-j| \neq 1, \tag{2.4}$$
$$t_{i+1} e_i t_{i+1} = t_i e_{i+1} t_i, \tag{2.5}$$
$$x_j t_i = t_i x_j, \qquad \text{if } |i-j| > 1, \tag{2.6}$$
$$x_{i+1} t_i = t_i x_i + e_i t_i, \tag{2.7}$$
$$e_i e_j = e_j e_i, \qquad \text{if } |i-j| > 1, \tag{2.8}$$
$$e_i x_j = x_j e_i, \qquad \text{if } |i-j| > 1, \tag{2.9}$$
$$x_i x_j = x_j x_i. \tag{2.10}$$

Let V and W be $\mathfrak{g}$-modules. Define

$$\begin{aligned} \mathrm{flip} : V \otimes W &\to W \otimes V, \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

The map flip is a $\mathfrak{g}$-module isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$.

Theorem 2.1.1. *Let M, V be $\mathfrak{g}$-modules. Define*

$$\begin{aligned} \partial\Phi : \partial\mathrm{Br}_k &\to \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes k}), \\ t_i &\mapsto \mathrm{id}_M \otimes \mathrm{id}_V^{\otimes i-1} \otimes \mathrm{flip} \otimes \mathrm{id}_V^{\otimes k-i-1}, \\ x_i &\mapsto \frac{1}{2}(\kappa_{M\otimes V^{\otimes i}} - \kappa_{M\otimes V^{\otimes i-1}} \otimes \mathrm{id}_V - \mathrm{id}_{M\otimes V^{\otimes i-1}} \otimes \kappa_V) \otimes \mathrm{id}_{V^{\otimes k-i}}, \\ e_i &\mapsto \mathrm{id}_M \otimes \mathrm{id}_{V^{\otimes i-1}} \otimes \frac{1}{2}(\kappa_{V\otimes V} - \kappa_V \otimes \mathrm{id}_V - \mathrm{id}_V \otimes \kappa_V) \otimes \mathrm{id}_{V^{\otimes k-i-1}}. \end{aligned}$$

Then $\partial\Phi$ defines an action of $\partial\mathrm{Br}_k$ on $M \otimes V^{\otimes k}$.

Proof. The relations (2.1), (2.2), (2.3), (2.6), (2.8), (2.9) and (2.10) are clear. It suffices to prove (2.4) and (2.5) for $k = 3$. Let $m \in M$ and let $v, v', v'' \in V$. Then

$$\partial\Phi(e_1)(m \otimes v \otimes v' \otimes v'') = \sum_i m \otimes b_i v \otimes b^i v' \otimes v''$$

and

$$\partial\Phi(e_2)(m \otimes v \otimes v' \otimes v'') = \sum_i m \otimes v \otimes b_i v' \otimes b^i v'',$$

where $\{b_i\}_i$ and $\{b^i\}_i$ are dual bases of $\mathfrak{g}$ with respect to $(\cdot|\cdot)$. It follows that

$$\begin{aligned} t_i e_j &= e_j t_i, \qquad \text{if } |i-j| \neq 1, \\ t_{i+1} e_i t_{i+1} &= t_i e_{i+1} t_i. \end{aligned}$$

These imply (2.4) and (2.5) respectively. Finally, it is sufficient to prove (2.7) for $k = 2$. We have

$$\begin{aligned}
\partial\Phi(x_2)(m \otimes v \otimes v') &= \frac{1}{2}(\Delta^2(\kappa) - \Delta(\kappa) \otimes 1 - 1 \otimes 1 \otimes \kappa)(m \otimes v \otimes v') \\
&= \frac{1}{2}(\Delta(\kappa \otimes 1 + 1 \otimes \kappa + 2\sum_i b_i \otimes b^i) - \Delta(\kappa) \otimes 1 - 1 \otimes 1 \otimes \kappa)(m \otimes v \otimes v') \\
&= (\sum_i \Delta(b_i) \otimes b^i)(m \otimes v \otimes v') \\
&= \sum_i (b_i m \otimes v \otimes b^i v' + m \otimes b_i v \otimes b^i v') \\
&= \partial\Phi(t_1 x_1 t_1 + e_1)(m \otimes v \otimes v').
\end{aligned}$$

This gives (2.7). □

Example 2.1.2. Consider the Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}(V)$. The defining representation V is n-dimensional and isomorphic to $L(\varepsilon_1)$. In particular, $\kappa_V = (\varepsilon_1|\varepsilon_1 + 2\rho) = 2n - 1$.

Let M be a $\mathfrak{gl}_n$-module. Let $E_{ij} \in \mathfrak{gl}_n$ denote the elementary matrix with 1 in te (i,j)-entry and 0 elsewhere. Then $\{E_{ij}\}_{i,j=1}^n$ and $\{E_{ji}\}_{i,j=1}^n$ are dual bases with respect to the trace form. Let $v, v' \in V$, then

$$\sum_{i,j=1}^n E_{ij} v \otimes E_{ji} v' = v' \otimes v.$$

We deduce that the element $e_i \in \partial\mathrm{Br}_k$ acts on $M \otimes V^{\otimes k}$ the same as t_i. The resulting algebra is the *degenerate affine Hecke algebra*, denoted $\partial H_k^{\mathrm{aff}}$. It is presented by generators $t_1, \ldots, t_{k-1}$, $x_1, \ldots, x_k$, subject to the relations

$$\begin{aligned}
t_i^2 &= 1, \\
t_i t_j &= t_j t_i, \qquad \text{if } |i-j| > 1, \\
t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, \\
t_i x_j &= x_j t_i, \qquad \text{if } |i-j| > 1, \\
x_{i+1} t_i &= t_i x_i + 1, \\
x_i x_j &= x_j x_i.
\end{aligned}$$

Let $\lambda \in P^+$, then

$$L(\lambda) \otimes L(\varepsilon_1) \simeq \bigoplus_{\lambda^+} L(\lambda^+),$$

where λ^+ runs through all weights $\mu \in P^+$ such that $\mu = \lambda + \varepsilon_i$ for some i. Thus, if $M = L(\lambda)$, $\lambda \in P^+$, then x_1 acts on the $L(\lambda + \varepsilon_i)$ isotypic component of $M \otimes V$ as multiplication by

$$\frac{1}{2}((\lambda + \varepsilon_i|\lambda + \varepsilon_i + 2\rho) - (\lambda|\lambda + 2\rho) - (\varepsilon_1|\varepsilon_1 + 2\rho)) = \lambda_i - i + 1,$$

where $\lambda_i = (\lambda|\varepsilon_i)$.

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