Determinants

Let V be a n-dimensional vector space over a field $\mathbb{F}$. Then $\bigwedge_{i=1}^{n} V$ is one dimensional. Let $A \in \mathrm{End}(V)$, then we have an induced endomorphism on $\bigwedge_{i=1}^{n} V$ given by $v_1 \wedge \cdots \wedge v_n \mapsto Av_1 \wedge \cdots \wedge Av_n$. As $\bigwedge_{i=1}^{n} V$ is one dimensional this induced map is some scalar k times the identity. Define the determinant of A, denoted $\det A$, to be this scalar k.

Prop Let $A, B \in \mathrm{End}(V)$, then $\det(AB) = \det A \cdot \det B$.

proof We compute:

$$\begin{aligned} \det(AB)(v_1 \wedge \cdots \wedge v_n) &= (AB)v_1 \wedge \cdots \wedge (AB)v_n \\ &= A(Bv_1) \wedge \cdots \wedge A(Bv_n) \\ &= \det B\,(Av_1 \wedge \cdots \wedge Av_n) \\ &= \det A \cdot \det B\,(v_1 \wedge \cdots \wedge v_n). \end{aligned}$$

Note that $\det(\mathrm{id}) = 1$. It follows that if A is invertible then $\det(A^{-1}) = (\det A)^{-1}$, in particular $\det A \neq 0$.

Prop Let $A \in \mathrm{End}(V)$, then $\det A \neq 0$ if and only if A is invertible.

proof The statement: if A is invertible, then $\det A \neq 0$ follows from the remarks above. For the converse, suppose $\det A \neq 0$. Fix a basis $e_1, \dots, e_n$ of V. Then $Ae_1 \wedge \cdots \wedge Ae_n \neq 0$. We infer that the vectors $Ae_1, \dots, Ae_n$ are linearly independent and thus form a basis for V. This is equivalent to A being invertible.

Prop Fix a basis $e_1, \dots, e_n$ of V and suppose $A \in \mathrm{End}(V)$ has matrix (a_{ij}) with respect to this basis. Then

$$\det A = \sum_{w \in S_n} (-1)^{\mathrm{sgn}(w)} a_{1w(1)} a_{2w(2)} \cdots a_{nw(n)},$$

where S_n is the group of permutations on $\{1, \dots, n\}$ and $\mathrm{sgn}(w)$ is the sign representation of S_n, i.e. maps even permutations to 1 and odd permutations to -1.

proof We compute:

$$\begin{aligned} Ae_1 \wedge \cdots \wedge Ae_n &= \sum_{i=1}^{n} a_{i1} e_i \wedge \cdots \wedge \sum_{i=1}^{n} a_{in} e_i \\ &= \sum_{w \in S_n} \left(a_{w(1)1} e_{w(1)} \wedge \cdots \wedge a_{w(n)n} e_{w(n)} \right) \\ &= \sum_{w \in S_n} \left(a_{w(1)1} \cdots a_{w(n)n} \right) \left(e_{w(1)} \wedge \cdots \wedge e_{w(n)} \right) \\ &= \sum_{w \in S_n} (-1)^{\mathrm{sgn}(w)} \left(a_{w(1)1} \cdots a_{w(n)n} \right) (e_1 \wedge \cdots \wedge e_n) \\ &= \sum_{w \in S_n} (-1)^{\mathrm{sgn}(w)} \left(a_{1w(1)} \cdots a_{nw(n)} \right) (e_1 \wedge \cdots \wedge e_n). \end{aligned}$$

So $\det A = \sum_{w \in S_n} (-1)^{\mathrm{sgn}(w)} a_{1w(1)} \cdots a_{nw(n)}$.